Afriat's Theorem and Graph

Runfa Hu

1 Introduction

1.1 Revealed Preference Theory

Economists study the optimal allocation of resources. However, before diving into that topic, we need to answer a basic question: what is a "better" allocation? For instance, how can you ensure that giving someone a graduate math textbook is better than giving them a ticket of Formula 1 Racing? What's your evidence?

A good starting point is to observe people's decisions. When I choose a cup of tea and leave the coffee still on the shelf, you can somehow infer that I "prefer" tea than coffee. This is an elementary idea of **Revealed Preference Theory**.

To be more rigorous and for the convenience of discussion, we're going to define "decision" and "preference" in the first section. Let X be an arbitrary set - you can imagine it to be all possible choices and status of the world.

Definition 1.1. Function $c : D \to 2^X \setminus \{\emptyset\}$ where $D \subseteq 2^X \setminus \{\emptyset\}$ is a *choice function* if $\forall B \in D,\ c(B) \subseteq B$.

You can imagine that B is a menu of elements from X, and $c(B)$ is the decision that we observe. We ask $c(B) \subseteq B$ in order to restrict people's decision within the menu, otherwise it will be meaningless.

Here the choice function c can be regarded as a "dataset." More specifically, every record of the data is a menu-decision pair $(B, c(B))$, and D is the collection of menus we've observed.

Remark 1.1. Excluding the empty set from the domain and image of c doesn't mean that we forbid people "not to choose anything." In fact, "not to choose anything" itself serves as a choice contained by X.

Remark 1.2. $c(\{a, b, c\}) = \{a, b\}$ doesn't mean we observe the agent choose both a and b simultaneously. In fact, if we allow he/she to choose multiple elements, those options should be contained by X as well. For instance, $c(\{a, b, \{a, b\}\}) = \{\{a, b\}\}$ describes that the agent choose the option of "having both a and b." What $c(\{a, b, c\}) = \{a, b\}$ really describes is that when facing the menu of $\{a, b, c\}$, the agent sometimes chooses a and sometimes chooses b.

Definition 1.2. Given a choice function $c : D \to 2^X \setminus \{\emptyset\}$. For any $x, y \in X$, x is *revealed preferred* to y, denoted by $x \succeq_c y$, if $\exists B \in D$ s.t. $x, y \in B$ and $x \in c(B)$.

This definition conveys the intuition that if someone chooses x while y is on the menu, we can tell that he/she "prefers" x to y. Note that here we didn't ask $y \notin c(B)$, and $x \succeq_c y$ is better interpreted as x is not "worse" than y.

Remark 1.3. The binary relation $\succeq_c$ is not necessarily transitive or complete, since we haven't make any further assumption on the choice function c.

1.2 Rationalizability

Since the choice function c is merely a dataset of agent's behaviors that we observed and the revealed preference relation is a description of the data without any structural assumptions, we can generally capture or model any behavior we want. However, a model describes everything is a model describes nothing - we can't make any prediction or have any useful insight into people's decision behaviors. Therefore, we're going to narrow our focus by introducing further assumptions.

An elementary idea of modeling people's decisions is to imagine a "happiness index" that somehow guides their behaviors - people try to maximize their feeling of happiness and avoid things that make them less happy. To be more rigorous, consider a *utility function* $u : X \to \mathbb{R}$ that describes such "happiness index."

Definition 1.3. A utility function $u : X \to \mathbb{R}$ *(weakly) rationalizes* the choice function c, or c is *(weakly) rationalizable* (by u), if $\forall B \in D$

$$c(B) \subseteq \arg\max_x \{u(x) \mid x \in B\}.$$

It seems unreasonable, since it describes a kind of people that attach a tag of real number to every object and status in the world and make decisions by comparing those numbers. However, we need to keep two ideas in mind:

- Like mathematicians' attitudes toward the Axiom of Choice in Set Theory, we should not stop making progress by judging whether the assumption is reasonable or not. We cares about what we can infer if accept it.
- Economists never think their models are true, and never assume that everyone's "greedily" maximizing their utility. In fact, they describes the condition under which people's behaviors could be construed as utility maximization. (We will show an if-and-only-if condition in the next section.)

Moreover, having a utility function is convenient for us to further study optimal allocation, from general equilibrium theory to mechanism design.

However, if we let $u(x) = 1,\ \forall x \in X$, it actually rationalizes every possible choice function, since for any c,

$$c(B) \subseteq B = \arg\max_x \{u(x) \mid x \in B\}.$$

To avoid this, we consider a particular class of utility functions. Let $X = \mathbb{R}^L_+$ (therefore X is equipped with topological structure), which can be regarded as the set of *consumption bundles*. Specifically, $(x_1, x_2, \cdots, x_L) \in \mathbb{R}^L_+$ means the option of having x_i units of good i, $\forall i \in [L]$. Moreover, if we have the price vector $(p_1, p_2, \cdots, p_L) \in \mathbb{R}^L_{++}$ and agent's wage w, then we can define his/her budget set as $B(p, w) := \{x \in X \mid \sum_{i=1}^L p_i x_i \leq w\}$ intuitively, which describes the consumption bundles that the agent can afford. Of course we can define a more general budget set by modifying our model.

Definition 1.4. A utility function $u : X \to \mathbb{R}$ is *locally non-satiated(LNS)* if $\forall x \in X,\ \epsilon > 0,\ \exists y \in X,$ s.t. $||y - x|| < \epsilon,\ u(y) > u(x)$.

Under this assumption, the agent who does utility maximization will not get "satisfied" on any interior point of X. For instance, a strictly monotone utility function is LNS. In fact, LNS is a weaker condition than monotonicity while ensuring the agent always choosing optimal bundles on the boundary of budget set. The following theorem formalizes this idea, but we skip the proof since it's elementary.

Proposition 1.1. Given a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the Walrasian demand (correspondence) $x : \mathbb{R}^n_{++} \times \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n_+$ is defined by

$$x(p, w) := \arg\max_{x \in \mathbb{R}^n_+} u(x) \quad \text{s.t. } p \cdot x \leq w.$$

If u is LNS, then $p \cdot x = w,\ \forall x \in x(p, w)$ ($x(p, w)$ satisfies Walras' Law).

As we discussed above, one of the reasons why we introduce LNS condition is to ensure our rationalizability problem being meaningful. Of course you can consider other assumptions, and the problem itself will change accordingly. Here we focus on LNS because it is weak, easy, and widely used in consumer theory. Therefore, it serves as a good starting point.

2 Afriat's Theorem

The first important question is: *When is a given dataset(choice function) rationalizable by an LNS utility function?*

In 1967, Afriat gave an if-and-only-if condition to this question. Although Afriat didn't prove it completely, researchers provided a wide variety of proofs based on his contribution. These proofs connected a wide range of fields and mathematical structures to the extent that they become more profound and intriguing than the theorem itself!

We first describe our question formally.

Given a finite data set $D := \{(p_i, x_i) \mid i \in [n]\} \subseteq \mathbb{R}^L_{++} \times \mathbb{R}^L_+$, we want to find a condition for D, under which there exists an LNS utility function u rationalizing it. That is to say,

$$x_i \in \arg\max_{x \in \mathbb{R}^L_+} u(x) \quad \text{s.t. } p_i \cdot x \leq p_i \cdot x_i,\ \forall i \in [n].$$

Define $a_{ij} := p_i(x_j - x_i),\ \forall i, j \in [n]$. If $a_{ij} \leq 0$, we have $x_j \in B(p_i, p_i \cdot x_i)$. Since we observe that x_i is chosen out from $B(p_i, p_i \cdot x_i)$, x_i is revealed preferred to x_j. Based on this idea, we introduce the Generalized Axiom of Revealed Preference (GARP) as the if-and-only-if condition for the above rationalizability test.

Definition 2.1. D satisfies the the Generalized Axiom of Revealed Preference (GARP) if for every ordered subset $\{i, j, k, \cdots, r\} \subseteq [n]$

$$\begin{matrix} a_{ij} \leq 0 & & a_{ij} = 0 \\ a_{jk} \leq 0 & & a_{jk} = 0 \\ \vdots & \text{implies} & \vdots \\ a_{ri} \leq 0 & & a_{ri} = 0 \end{matrix}$$

Theorem 2.1 (Afriat's Theorem)**.** The data set D satisfies GARP if and only if there exists an LNS utility function that rationalizes D.

We provide an elementary graphic proof of Afriat's Theorem in the next section.

3 Graphic Proof of Afriat's Theorem

The "if" part of the proof is elementary. We focus on the "only if" part.

The proof is divided into two steps:

- Step 1: If D satisfies GARP, then there exists two sets of numbers $\{\phi_i\}_{i=1}^n \subset \mathbb{R}$ and $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_{++}$ that satisfy "Afriat inequalities."
- Step 2: Construct an LNS utility function that rationalizes D based on $\{\phi_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$.

Step 2 is Afriat's original argument. We first review his construction, then finish the proof by completing step 1.

....

Proof of Step 2

$\{\phi_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ satisfy Afriat inequalities if they serve as a solution to the following system:

$$\phi_j \leq \phi_i + \lambda_i a_{ij},\ \forall i, j \in [n].$$

If such ϕ_i and λ_i exist, then we can construct the utility function as follow:

$$u(x) := \min_{i\in[n]}\{\phi_i + \lambda_i p_i(x - x_i)\}.$$

1. u is strictly monotone, thus LNS. This is because $\lambda_i > 0,\ p_i > 0,\ \forall i \in [n]$. Therefore u is the pointwise minimum of n strictly increasing linear functions. In fact, u is also concave.
2. $u(x_j) = \phi_j,\ \forall j \in [n]$. This is because $u(x_j) = \min_{i\in[n]}\{\phi_i + \lambda_i p_i(x_j - x_i)\} = \min_{i\in[n]}\{\phi_i + \lambda_i a_{ij}\} = \phi_j + \lambda_i a_{jj} = \phi_j$.
3. $x_i \in \arg\max_{x\in\mathbb{R}_+^L} u(x) \quad \text{s.t. } p_i x \leq p_i x_i,\ \forall i \in [n]$. This is because for all $x \in \mathbb{R}_+^L$ such that $p_i x \leq p_i x_i$, we have $p_i(x - x_i) \leq 0$, thus $u(x) \leq \phi_i + \lambda_i p_i(x - x_i) \leq \phi_i = u(x_i)$.

Therefore, u is the utility function we want.

Now, we only need to show the existence of $\{\phi_i\}_{i=1}^n$ and $\{\lambda_i\}_{i=1}^n$ given GARP holds for D (step 1).

....

Proof of Step 1

Consider a weighted directed graph $G = (V, A)$, where $V = [n],\ A = V \times V$. For every arc $(i, j) \in A$, it is weighted by a_{ij}.

Lemma 3.1. The following three statements are equivalent:

(i) D satisfies GARP;

(ii) For any cycle $i \to j \to k \to \cdots \to r \to i$ in graph G, if every arc is non-positively weighted, then they're actually zero-weighted;

(iii) For any closed walk $i \to j \to k \to \cdots \to r \to i$ in graph G, if every arc is non-positively weighted, then they're actually zero-weighted.

Proof. (ii) is actually a restatement of the definition of GARP in graph G, therefore (i) $\Leftrightarrow$ (ii). Since (iii) is clearly stronger than (ii), we only need to show (ii) $\Rightarrow$ (iii).

For any non-positively weighted closed walk $i \to j \to k \to \cdots \to r \to i$, there exists repeated vertices in this sequence (e.g. i at the beginning and the end). Take a pair of repeated vertices (s, s) such that the closed sub-walk $s \to t \to \cdots \to s$ has the minimal length of all other closed sub-walks. Therefore, there's no repeated vertices within $s \to t \to \cdots \to s$. As a result, the closed sub-walk with minimal length is actually a non-positively weighted cycle. According to (ii), it is zero-weighted. We can construct a new closed walk by deleting this zero-weighted sub-walk:

$$\text{from } i \to \cdots \to p \to s \to t \to \cdots \to s \to q \to \cdots \to i$$

$$\text{to } i \to \cdots \to p \to s \to q \to \cdots \to i.$$

By this deleting process, the total length of the closed walk is strictly decreased, and the new closed walk is still non-positively weighted and still exists repeated vertices. Therefore, we can continue this process until i is the only repeated vertex and the remaining closed walk is actually a non-positively weighted cycle, which is again zero-weighted according to (ii). Finally, during the process, every arc in the original closed walk is forced to be zero-weighted, which is exactly (iii). □

Define a binary relation $\sim$ on V. $\forall a, b \in V$, $a \sim b$ if there exists a zero-weighted closed walk containing both a and b. In fact, $\sim$ is an equivalence relation.

1. $\sim$ is reflexive: $\forall i \in V$, $a_{ii} = 0$, thus $i \to i$ is a zero-weighted closed walk containing i, which means $i \sim i$.

2. $\sim$ is symmetric: $\forall i, j \in V$, if $i \sim j$, then the zero-weighted closed walk containing i and j also implies $j \sim i$.

3. $\sim$ is transitive: $\forall i, j, k \in V$, if $i \sim j$ and $j \sim k$, then there exists zero-weighted walks from i to j, from j to k, from k to j and from j to i, respectively. We can construct a zero-weighted closed walk containing i and k by combining these four zero-weighted walks, therefore $i \sim k$.

Since $\sim$ is an equivalence relation, it forms an equivalence class partitioning on V. Define $\tilde{V} := \{[i] \mid \forall i \in V\}$, where $[i] = \{j \in V \mid j \sim i\}$, and $\tilde{A} \subseteq \tilde{V} \times \tilde{V}$ such that $\forall [i], [j] \in \tilde{V}$, when $[i] \neq [j]$, $([i], [j]) \in \tilde{A}$ if and only if $\min_{s \in [i], t \in [j]} a_{st} \leq 0$; when $[i] = [j]$, $([i], [j]) \in \tilde{A}$ if and only if $\min_{s \in [i], t \in [j]} a_{st} < 0$. Let $\tilde{G} = (\tilde{V}, \tilde{A})$ be a directed graph.

Theorem 3.1. D satisfies GARP if and only if $\tilde{G}$ contains no cycle.

4 $O(n^2)$ Graphic Algorithm for Testing GARP

5 Cover Choice Data with Preferences