Integer Formulation for the Houtman-Maks Index

Runfa Hu

ACEM, Shanghai Jiao Tong University

May 2026

Outline

Classical Settings in Consumer Theory

Consider a consumer making the following purchase decision.

- $x \in \mathbb{R}^L_+$ denotes a consumption bundle (a "basket" of goods) over L different goods.
- $p = (p_1, \cdots, p_L) \in \mathbb{R}^L_{++}$ denotes the price vector.
- $w \in \mathbb{R}_+$ denotes the wealth.
- Assume that the consumer has a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$.
- The consumer chooses $x(p, w)$ by solving:

$$x(p, w) := \arg\max_{x \in \mathbb{R}^L_+} u(x) \quad \text{s.t. } \langle p, x \rangle \leq w.$$

Classical Settings in Consumer Theory

Consider a consumer making the following purchase decision.

- $x \in \mathbb{R}_+^L$ denotes a consumption bundle (a "basket" of goods) over L different goods.
- $p = (p_1, \cdots, p_L) \in \mathbb{R}_{++}^L$ denotes the price vector.
- $w \in \mathbb{R}_+$ denotes the wealth.
- Assume that the consumer has a utility function $u : \mathbb{R}_+^L \to \mathbb{R}$.
- The consumer chooses $x(p, w)$ by solving:

$$x(p, w) := \arg\max_{x \in \mathbb{R}_+^L} u(x) \quad \text{s.t. } \langle p, x \rangle \leq w.$$

- Why utility function?

An Inverse Problem (Samuelson, 1938)

- Suppose we observe a dataset of consumer decisions $D := \{(p_i, q_i) \mid i \in [n]\} \subset \mathbb{R}^L_{++} \times \mathbb{R}^L_{+}$.
- $p_i = (p_i^1, p_i^2, \cdots, p_i^L)$ records the price of L products in the i-th observation.
- $q_i = (q_i^1, q_i^2, \cdots, q_i^L)$ records the amount of each product chosen by the consumer when facing p_i
- Can D be explained by the maximization of a well-behaved utility function?

An Inverse Problem (Samuelson, 1938)

This problem can be rigorously defined as follows:

Definition (Rationalizablility)

Given a concave and strictly monotone utility function $u : \mathbb{R}_+^L \to \mathbb{R}$, the dataset D is *rationalizable (by u)* if for each $i \in [n]$

$$q_i \in \arg\max_{q \in \mathbb{R}_+^L} u(q) \quad \text{s.t. } \langle p_i, q \rangle \leq \langle p_i, q_i \rangle.$$

An Inverse Problem (Samuelson, 1938)

This problem can be rigorously defined as follows:

Definition (Rationalizablility)

Given a concave and strictly monotone utility function $u : \mathbb{R}_+^L \to \mathbb{R}$, the dataset D is *rationalizable (by u)* if for each $i \in [n]$

$$q_i \in \arg\max_{q \in \mathbb{R}_+^L} u(q) \quad \text{s.t. } \langle p_i, q \rangle \leq \langle p_i, q_i \rangle.$$

- When is D rationalizable?
- (Afriat, 1967) provided an if-and-only-if condition.

Afriat's Theorem

- $a_{ij} := \langle p_i, q_j - q_i \rangle, \ \forall i, j \in [n]$.

Definition (GARP)

D satisfies the Generalized Axiom of Revealed Preference (GARP) if for every ordered subset $\{i, j, k, \cdots, r\} \subseteq [n]$

$$\begin{array}{ccc} a_{ij} \leq 0 & & a_{ij} = 0 \\ a_{jk} \leq 0 & & a_{jk} = 0 \\ \vdots & \text{implies} & \vdots \\ a_{ri} \leq 0 & & a_{ri} = 0 \end{array}$$

Theorem (Afriat's Theorem)

D is rationalizable if and only if D satisfies GARP.

Houtman-Maks Index

- (Talla Nobibon et al., 2015) and (Shiozawa, 2016) provided $O(n^2)$ algorithms for testing rationalizability.
- We need tools to measure the severity of violations of rationalizability.
- Houtman-Maks Index is one of such measures.

$$
\begin{aligned}
HM(D) := \max_{X \subseteq D} \quad & \frac{|X|}{|D|} \\
\text{s.t.} \quad & X \text{ is rationalizable}
\end{aligned}
$$

Houtman-Maks Index

- (Talla Nobibon et al., 2015) and (Shiozawa, 2016) provided $O(n^2)$ algorithms for testing rationalizability.
- We need tools to measure the severity of violations of rationalizability.
- Houtman-Maks Index is one of such measures.

$$\begin{aligned} HM(D) := \max_{X \subseteq D} \quad & \frac{|X|}{|D|} \\ \text{s.t.} \quad & X \text{ is rationalizable} \end{aligned}$$

- (Houtman and Maks, 1985) established connections between their index and the feedback vertex set problem.
- (Smeulders et al., 2014) proved its NP-hardness.
  - unless P=NP, there is no polynomial time algorithm to solve this index.

Outline

Integer Programming

Consider an optimization problem:

Mixed Integer Linear Program (MILP)

$$\begin{aligned} \max \quad & \langle c, x \rangle + \langle h, y \rangle \\ \text{s.t.} \quad & x \in \mathbb{Z}^n, y \in \mathbb{R}^p \\ & Ax + Gy \leq b \end{aligned}$$

- c, h, A, G, b are data of the problem.
- x, y satisfying the constraints are feasible solutions.
- We want to find feasible solutions that maximize $\langle c, x \rangle + \langle h, y \rangle$.
- Our goal is to formulate NP-hard problems into MILP, which can be solved by optimization solvers.

Outline

Sign Vector and Hypersum

- $\forall\ \mathcal{S} \subseteq \{0,+,-\}^{[n]}$, we define their hypersum as:

$$\boxplus_{X\in\mathcal{S}} X := \left\{ Y \in \{0,+,-\}^{[n]} \;\middle|\; Y(i) \in \boxplus_{X\in\mathcal{S}} X(i),\ \forall i \in [n] \right\}.$$

- The span of $\mathcal{S}$ is defined by

$$\mathrm{span}(\mathcal{S}) := \bigcup_{\mathcal{X}\subseteq\mathcal{S}} \left\{ \boxplus_{X\in\mathcal{X}} X \right\}.$$

- It is clear that $\mathrm{span}(\mathcal{S})$ is closed under the hypersum operation.
- It is the smallest set of sign vectors containing $\mathcal{S}$ that is closed under the hypersum operation (intuitively "hypersum-hull").

Sign Vector as Orthant

- For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $\mathrm{sign}(x)$ denotes the sign vector:

$$(\mathrm{sign}(x_1), \mathrm{sign}(x_2), \cdots, \mathrm{sign}(x_n)) \in \{0, +, -\}^{[n]}.$$

- For $S \subseteq \mathbb{R}^n$, $\mathrm{sign}(S) := \bigcup_{x \in S} \mathrm{sign}(x)$.
- For $X \in \{0, +, -\}^{[n]}$, the orthant in $\mathbb{R}^n$ corresponding to X is $\mathrm{sign}^{-1}(X) := \{x \in \mathbb{R}^n \mid \mathrm{sign}(x) = X\}$.
- For $\mathcal{S} \subseteq \{0, +, -\}^{[n]}$, $\mathrm{sign}^{-1}(\mathcal{S}) := \bigcup_{X \in \mathcal{S}} \mathrm{sign}^{-1}(X)$.

Hypersum Span and Convex Hull

We have the following two simple but useful arguments:

Lemma

$\forall \mathcal{S} \subseteq \{0,+,-\}^{[n]},\ \mathrm{conv}(\mathrm{sign}^{-1}(\mathcal{S})) = \mathrm{sign}^{-1}(\mathrm{span}(\mathcal{S})).$

- The hypersum span of sign vectors is the convex hull of their corresponding orthants.

Hypersum Span and Convex Hull

We have the following two simple but useful arguments:

Lemma

$\forall \mathcal{S} \subseteq \{0, +, -\}^{[n]}, \ \mathrm{conv}(\mathrm{sign}^{-1}(\mathcal{S})) = \mathrm{sign}^{-1}(\mathrm{span}(\mathcal{S}))$.

- The hypersum span of sign vectors is the convex hull of their corresponding orthants.

Corollary

$\forall S \subseteq \mathbb{R}^n$, $\mathrm{sign}^{-1}(\mathrm{span}(\mathrm{sign}(S)))$ *is the tightest convex relaxation of* S *that consists of orthants in* $\mathbb{R}^n$.

- We call $\mathrm{sign}^{-1}(\mathrm{span}(\mathrm{sign}(S)))$ the orthant envelope of S, denoted by $\mathrm{orth}(S)$.

Outline

Steps of Constructing Formulation

$$
\begin{aligned}
HM(D) := \max_{X \subseteq D} \quad & \frac{|X|}{|D|} \\
\text{s.t.} \quad & X \text{ is rationalizable}
\end{aligned}
$$

- Step 1: Characterize rationalizablility.
- Step 2: Formulate subset selection $X \subseteq D$.
- Step 3: Convexification.

Step 1.0: Afriat's Inequalities

- What we have: $D = \{(p_i, q_i) \mid i \in [n]\}$.
- $a_{ij} = \langle p_i, q_j - q_i \rangle, \ \forall i, j \in [n]$ and $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

Theorem (Afriat's Inequalities)

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \lambda_i a_{ij} \geq 0 \quad \forall i, j \in [n]$$

$$\lambda_i > 0 \quad \forall i \in [n].$$

Step 1.1: Combinatorial Version of Afriat's Inequalities

Theorem (Extension)

For all $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ such that $\text{sign}(B) = \text{sign}(A)$*,*

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \lambda_i b_{ij} \geq 0 \quad \forall i, j \in [n]$$

$$\lambda_i > 0 \quad \forall i \in [n].$$

- The above system of linear inequalities is defined as $AI(B)$.
- Value-independent: only the sign of a_{ij} matters.

Step 1.2: Ideas of Proof - Cycle Polytope

Let $G_A = (V, E, a)$ be a complete directed graph:

- $V = [n]$, $E = V \times V$.
- Each $(i, j) \in E$ is weighted by a_{ij}.

For each cycle $C \subset G_A$, define $a^C \in \mathbb{R}^n$ as follows:

$$a_i^C := \begin{cases} a_{iC(i)} & \text{if } i \in V(C) \\ 0 & \text{otherwise} \end{cases} \quad , \forall i \in V.$$

- Let $P_A := \{a^C \mid \forall \text{ cycle } C \subset G\}$.
- $\text{conv}(P_A)$ is the cycle polytope of G_A.

Step 1.3: Ideas of Proof

- The following five arguments are equivalent:
  1. D is rationalizable (i.e. $HM(D) = 1$);
  2. D satisfies GARP;
  3. $\mathbb{R}^n_- \bigcap \text{orth}(P_A) = \{0\}$;
  4. For all B such that $\text{sign}(B) = \text{sign}(A)$, $\mathbb{R}^n_- \bigcap \text{conv}(P_B) = \{0\}$;
  5. For all B such that $\text{sign}(B) = \text{sign}(A)$, $AI(B)$ is feasible.
- $\{AI(B) \mid B \in \mathbb{R}^{n \times n},\ \text{sign}(B) = \text{sign}(A)\}$ are simultaneously feasible or infeasible.

Step 1.4: Ideas of Proof - 1 $\Rightarrow$ 2

1 D is rationalizable (i.e. $HM(D) = 1$);

2 D satisfies GARP.

- Suppose D is rationalized by a strictly monotone utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$: for each $i \in [n]$

$$q_i \in \arg\max_{q \in \mathbb{R}^L_+} u(q) \quad \text{s.t. } \langle p_i, q \rangle \leq \langle p_i, q_i \rangle.$$

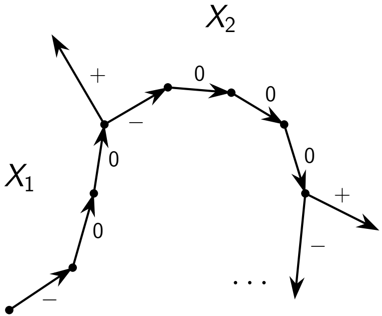
- $a_{ij} \leq 0 \Rightarrow \langle p_i, q_j - q_i \rangle \leq 0 \Rightarrow \langle p_i, q_j \rangle \leq \langle p_i, q_i \rangle \Rightarrow u(q_j) \leq u(q_i)$.
- $\forall C \subset G_A$, if $a_{iC(i)} \leq 0 \ \forall i \in V(C)$, then $u(q_i) = u(q_{C(i)})$.
- Since u is strictly monotone, $u(q_i) = u(q_j) \Rightarrow a_{ij} \geq 0$.
- Therefore, $a_{iC(i)} = 0 \ \forall i \in V(C)$.

Step 1.4: Ideas of Proof - 2 $\Rightarrow$ 3

2 D satisfies GARP;

3 $\mathbb{R}^n_- \bigcap \mathrm{orth}(P_A) = \{0\}$.

- Recall that $\mathrm{orth}(P_A) = \mathrm{sign}^{-1}(\mathrm{span}(\mathrm{sign}(P_A)))$.
- Otherwise, if there exists $X \in \mathrm{span}(\mathrm{sign}(P_A))$ such that $X^- \neq \emptyset$.
- Suppose $X = X_1 \boxplus X_2 \boxplus \cdots \boxplus X_k$, each $X_i \in \mathrm{sign}(P_A)$ corresponds to a cycle in G_A.

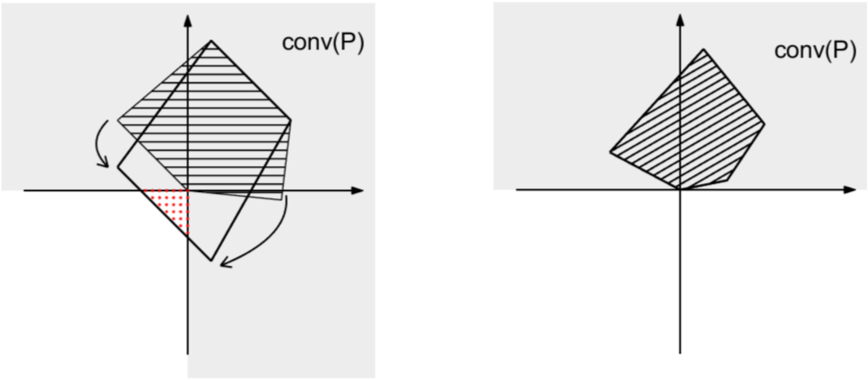


Step 1.4: Ideas of Proof - 3 $\Rightarrow$ 4

3 $\mathbb{R}^n_- \bigcap \text{orth}(P_A) = \{0\}$

4 For all B such that $\text{sign}(B) = \text{sign}(A)$, $\mathbb{R}^n_- \bigcap \text{conv}(P_B) = \{0\}$.

- We can "move" the vertices of P freely within each orthant!
- Rigidity of the cycle polytope.



Step 1.4: Ideas of Proof - (4) $\Rightarrow$ (5)

(4) For all B such that $\text{sign}(B) = \text{sign}(A)$, $\mathbb{R}^n_- \bigcap \text{conv}(P_B) = \{0\}$;

(5) For all B such that $\text{sign}(B) = \text{sign}(A)$, $AI(B)$ is feasible.

- Since $\mathbb{R}^n_- \bigcap \text{conv}(P_B) = \{0\}$, by hyperplane separation theorem, $\exists \lambda \in \mathbb{R}^n_{++}$ s.t. for each $b^C \in P_B$ corresponding to cycle $C \subset G_B$

$$\langle \lambda, b^C \rangle \geq 0 \Leftrightarrow \sum_{i \in V(C)} \lambda_i b_{iC(i)} \geq 0.$$

- Consider a complete directed graph $G = (V, E, w)$, such that $V = [n]$, $E = V \times V$, and each $(i, j) \in E$ is weighted by $w_{ij} = \lambda_i b_{ij}$.
- G contains no cycles with negative length!

Step 1.4: Ideas of Proof - ④ $\Rightarrow$ ⑤

④ For all B such that $\mathrm{sign}(B) = \mathrm{sign}(A)$, $\mathbb{R}^n_- \bigcap \mathrm{conv}(P_B) = \{0\}$;

⑤ For all B such that $\mathrm{sign}(B) = \mathrm{sign}(A)$, $AI(B)$ is feasible.

- Given $\lambda \in \mathbb{R}^n_{++}$, does $\phi \in \mathbb{R}^n$ exist?

$$\phi \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n_{++}$$

$$\phi_j - \phi_i \le \lambda_i b_{ij} \quad \forall i, j \in [n]$$

Theorem (Negative Cycle)

$\phi \in \mathbb{R}^n$, $\phi_j - \phi_i \le w_{ij}$ $\forall i, j \in [n]$ *is feasible if and only if* $G = (V, E, w)$ *contains no cycles with negative length.*

- Network flow: the dual of shortest path problem.
- By the construction of λ, such ϕ exists.

Step 1.4: Ideas of Proof - 5 $\Rightarrow$ 1

5 For all B such that $\text{sign}(B) = \text{sign}(A)$, $AI(B)$ is feasible;

1 D is rationalizable (i.e. $HM(D) = 1$).

Take $B = A$, we have the following Afriat's inequalities feasible:

$$\phi \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n_{++}$$

$$\phi_j - \phi_i \leq \lambda_i a_{ij} \quad \forall i, j \in [n]$$

Therefore, for each $i, j \in [n]$, we have $\phi_j \leq \phi_i + \lambda_i \langle p_i, q_j - q_i \rangle$.

$$u(x) = \min_{i \in [n]} \{\phi_i + \lambda_i \langle p_i, x - q_i \rangle\}$$

$u : \mathbb{R}^L_+ \to \mathbb{R}$ is a concave and strictly monotone utility function that rationalizes D.

Step 1.5: Ideas of Proof

- The following five arguments are equivalent:
  1. D is rationalizable (i.e. $HM(D) = 1$);
  2. D satisfies GARP;
  3. $\mathbb{R}^n_- \bigcap \text{orth}(P_A) = \{0\}$;
  4. For all B such that $\text{sign}(B) = \text{sign}(A)$, $\mathbb{R}^n_- \bigcap \text{conv}(P_B) = \{0\}$;
  5. For all B such that $\text{sign}(B) = \text{sign}(A)$, $AI(B)$ is feasible.
- $\{AI(B) \mid B \in \mathbb{R}^{n \times n},\ \text{sign}(B) = \text{sign}(A)\}$ are simultaneously feasible or infeasible.

Step 1.5: Ideas of Proof

- The following five arguments are equivalent:
  1. D is rationalizable (i.e. $HM(D) = 1$);
  2. D satisfies GARP;
  3. $\mathbb{R}^n_- \bigcap \mathrm{orth}(P_A) = \{0\}$;
  4. For all B such that $\mathrm{sign}(B) = \mathrm{sign}(A)$, $\mathbb{R}^n_- \bigcap \mathrm{conv}(P_B) = \{0\}$;
  5. For all B such that $\mathrm{sign}(B) = \mathrm{sign}(A)$, $AI(B)$ is feasible.
- $\{AI(B) \mid B \in \mathbb{R}^{n \times n},\ \mathrm{sign}(B) = \mathrm{sign}(A)\}$ are simultaneously feasible or infeasible.
  - $\{P_B \mid B \in \mathbb{R}^{n \times n},\ \mathrm{sign}(B) = \mathrm{sign}(A)\}$ share the same orthant!
  - Rigidity of the cycle polytope.

Step 2.1: Clever Choice of (b_{ij})

Theorem (Extension)

For all $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ *such that* $\operatorname{sign}(B) = \operatorname{sign}(A)$,

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n,\ \ \lambda \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \lambda_i b_{ij} \geq 0 \quad \forall i, j \in [n]$$

$$\lambda_i > 0 \quad \forall i \in [n].$$

Step 2.1: Clever Choice of (b_{ij})

Theorem (Extension)

For all $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ such that $\text{sign}(B) = \text{sign}(A)$,

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \lambda_i b_{ij} \geq 0 \quad \forall i, j \in [n]$$

$$\lambda_i \geq 1 \quad \forall i \in [n].$$

Step 2.1: Clever Choice of (b_{ij})

Theorem (Extension)

For all $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ such that $\mathrm{sign}(B) = \mathrm{sign}(A)$,

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n,\ \ \lambda \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \lambda_i b_{ij} \geq 0 \quad \forall i, j \in [n]$$

$$W \geq \lambda_i \geq 1 \quad \forall i \in [n].$$

Step 2.1: Clever Choice of (b_{ij})

Theorem (Extension)

For all $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ such that $\mathrm{sign}(B) = \mathrm{sign}(A)$,

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \lambda_i b_{ij} \geq 0 \quad \forall i, j \in [n]$$

$$W \geq \lambda_i \geq 1 \quad \forall i \in [n].$$

- Since (b_{ij}) can be chosen freely...
- Can we choose (b_{ij}) cleverly, so that we can control W?
- Surprisingly, $W = 1$.

Step 2.1: Clever Choice of (b_{ij})

Theorem (Extension)

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n$$

$$\phi_i - \phi_j + \tilde{a}_{ij} \geq 0 \quad \forall i, j \in [n]$$

- Since (b_{ij}) can be chosen freely...
- Can we choose (b_{ij}) cleverly, so that we can control W?
- Surprisingly, $W = 1$.

Step 2.1: Clever Choice of (b_{ij})

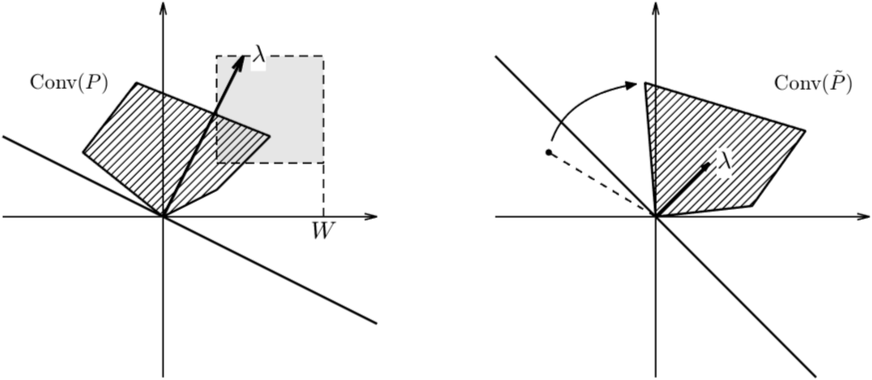


Figure: Moving conv(P) to control W

Step 2.2: Eliminate λ

Theorem (Extension)

D is rationalizable if and only if the following system of linear inequalities is feasible:

$$\phi \in \mathbb{R}^n$$
$$\phi_i - \phi_j + \tilde{a}_{ij} \geq 0 \quad \forall i, j \in [n]$$

For each $i, j \in [n]$

$$\tilde{a}_{ij} = \begin{cases} M & \text{if } \operatorname{sign}(a_{ij}) = + \\ -m & \text{if } \operatorname{sign}(a_{ij}) = - \\ 0 & \text{otherwise.} \end{cases}$$

$M > 0$ sufficiently large, $m > 0$ sufficiently small.

Step 2.3: Subset Selection

- Definition of Houtman-Maks index:

$$HM(D) := \max_{X \subseteq D} \quad \frac{|X|}{|D|}$$
$$\text{s.t.} \quad X \text{ is rationalizable}$$

- A simple non-linear formulation:

$$\begin{aligned} \max \quad & \frac{\sum_{i=1}^{n} s_i}{n} \\ \text{s.t.} \quad & s \in \{0,1\}^n,\ \phi \in \mathbb{R}^n \\ & s_i s_j (\phi_i - \phi_j + \tilde{a}_{ij}) \geq 0,\ \forall i,j \in [n]. \end{aligned}$$

Step 2.4: Simplification

- Definition of Houtman-Maks index:

$$HM(D) := \max_{X \subseteq D} \quad \frac{|X|}{|D|}$$
$$\text{s.t.} \quad X \text{ is rationalizable}$$

- A simple non-linear formulation:

$$\max \quad \frac{\sum_{i=1}^{n} s_i}{n}$$
$$\text{s.t.} \quad s \in \{0,1\}^n,\ \phi \in \mathbb{R}^n$$
$$\boxed{s_i(\phi_i - \phi_j + \tilde{a}_{ij}) \geq 0,\ \forall i,j \in [n]}\,.$$

Step 2.5: Simplification

- Definition of Houtman-Maks index:

$$HM(D) := \max_{X \subseteq D} \quad \frac{|X|}{|D|}$$
$$\text{s.t.} \quad X \text{ is rationalizable}$$

- A simple non-linear formulation:

$$\max \quad \frac{\sum_{i=1}^{n} s_i}{n}$$
$$\text{s.t.} \quad s \in \{0,1\}^n, \ \phi \in [0,1]^n$$
$$s_i(\phi_i - \phi_j + \tilde{a}_{ij}) \geq 0, \ \forall i,j \in [n].$$

Step 3: Convexification

- Definition of Houtman-Maks index:

$$HM(D) := \max_{X \subseteq D} \quad \frac{|X|}{|D|}$$
$$\text{s.t.} \quad X \text{ is rationalizable}$$

- A simple linear formulation:

$$\begin{aligned}
\max \quad & \frac{\sum_{i=1}^{n} s_i}{n} \\
\text{s.t.} \quad & s \in \{0,1\}^n,\ \phi \in [0,1]^n \\
& \phi_i \geq (1 - \tilde{a}_{ij}) s_i + \phi_j - 1,\ \forall i, j \in [n] \\
& \phi_i \geq -\tilde{a}_{ij} s_i,\ \forall i, j \in [n].
\end{aligned}$$

The State-of-art Formulation

- (Demuynck and Rehbeck, *Economic Theory*, 2023) provided the following state-of-art formulation:

$$
\begin{aligned}
\max \quad & \frac{\sum_{i=1}^{n} s_i}{n} \\
\text{s.t.} \quad & s \in \{0,1\}^n,\ \phi \in [0,1]^n \text{ and } U \in \{0,1\}^{n\times n} \\
& \phi_i - \phi_j \le -\epsilon + 2U_{i,j} \quad \forall i,j \in [n] \\
& U_{i,j} - 1 \le \phi_i - \phi_j \quad \forall i,j \in [n] \\
& -a_{ij} \le -\delta + \alpha[U_{i,j} + (1 - s_i)] \quad \forall i,j \in [n] \\
& \alpha(U_{j,i} + s_i - 2) \le a_{ij} \quad \forall i.j \in [n]
\end{aligned}
$$

Numerical Experiment

n	HM Index	Running Time	
		OR	ET
100	0.83	0.24s	0.76s
100	0.9	0.22s	1.06s
200	0.77	8.42s	26.72s
200	0.835	3.75s	19.04s
500	0.66	13.68min	9.27h
500	0.76	7.68min	18.67min
500	0.844	2.95min	22.78min
500	0.916	53s	10.42min
1000	0.855	39min	8.02h
2000	0.908	1.70h	/

Table: Running Time Comparison