Continuous Ramsey Model under Population Decline

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Abstract

A Ramsey model under the assumption of population decline is established. I first considered a standard discrete Ramsey model and introduced population growth rate into the model. After solving the first order condition, I switched the time-series equations into continuous ones and derived a two-dimension ODE system from it. Then parameters in the system are calibrated. I did a numerical simulation on MATLAB based on the calibration and analyzed the phase diagram under different economic settings, especially population decline, using a shooting algorithm.

Keywords: Ramsey Model, population decline, ODE system, numerical simulation

Introduction

In macroeconomics , many growth models based on different assumptions are trying to explain and understand the long-term growth of an economy. One of the most classic one is Solow model, which is generally the first step into macroeconomics for most students. However, Solow model does not contain any optimization, which is important in economics theory. Fortunately, the neoclassical growth model improves this critical point. And it is the most popular framework for modern macroeconomic analysis[1].

Since the neoclassical model has three variables, say consumption per capita (c_t), capital stock per capita (k_t) and labor hours (l_t), which makes it hard to do 2-D phase diagram analysis. Here we focus on a simplified version of neoclassical growth model – the Ramsey model.

The Ramsey model sets household labor hours to constant $(l_t = 1)$, thus the optimization problem fully relys on consumption and capital stock. There are two benefits of this simplification. On the one hand, the first order condition of the problem becomes easier to calculate and understand. On the other hand, we can "label" the status of economy within two variables, say a two-dimension vector, and the ODE system derived from the problem has two equations as well, which is comfortable for us to imagine in a consumption – capital stock coordinate system.

Model Settings & Theoretical Analysis

The Ramsey model is described as follow:

$$\max U_0 = \sum_{t=0}^{+\infty} \beta^t \frac{c_t^{1-1/\theta} - 1}{1 - 1/\theta}$$

s.t.

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}$$

$$K_{t+1} = (1-\delta)K_t + I_t$$

$$N_{t+1} = (1+\eta)N_t,\ \ L_t = N_t l_t$$

$$Y_t = C_t + I_t$$

and

$$y_t = \frac{Y_t}{N_t},\ \ k_t = \frac{K_t}{N_t},\ \ c_t = \frac{C_t}{N_t}\ \ , i_t = \frac{I_t}{N_t}$$

$$l_t = 1,\ \ c_t \geq 0,\ \ k_{t+1} \geq 0$$

while

$$k_0 > 0\ \ is\ \ given$$

$Y_t, K_t, L_t, I_t, C_t, N_t$ refer to real GDP, real capital stock, labor supply, real investment, real consumption, population at time t, respectively. The variables divided by N_t, which are lowercase $(y_t,\ k_t,\ c_t,\ i_t)$ above, mean the per capita form of each economic concept respectively. δ is the capital depreciation rate. η is the population growth rate. β is the discount factor of household utility.

l_t between 0 and 1 is labor hours (proportion of work within a single day). Here in the Ramsey model, we set $l_t = 1$, which means household in the economy always chooses to work all day long without any break. It seems to be a strong assumption that is far different from reality, but it's useful and does not affect the main ideas of the model.

Proposition 1. The Ramsey model above can be approximately described by a two-dimension ODE system:

$$\frac{dc}{dt} = \theta[ln\beta - \ln(1+\eta) + A\alpha k^{\alpha-1} - \delta]$$

$$\frac{dk}{dt} = \frac{1}{1+\eta}[Ak^{\alpha} - c - (\eta+\delta)k]$$

$$given\ \ \mathrm{c}_0\ \ and\ \ k_0$$

Proof:

To solve the model, I first switch all variables into per capita form and combine all equations into a status-transferring equation.

$$l_t = 1$$

$\Rightarrow$ $$L_t = N_t l_t = N_t$$

$\Rightarrow$ $$Y_t = AK_t^{\alpha} N_t^{1-\alpha}$$

$\Rightarrow$ $$\frac{Y_t}{N_t} = A(\frac{K_t}{N_t})^{\alpha}$$

$\Rightarrow$ $$y_t = Ak_t{}^{\alpha}$$

On the other hand,

$$\frac{K_{t+1}}{N_{t+1}} \cdot \frac{N_{t+1}}{N_t} = (1-\delta)\frac{K_t}{N_t} + \frac{I_t}{N_t}$$

$\Rightarrow$ $$k_{t+1}(1+\eta) = (1-\delta)k_t + i_t$$

Meanwhile,

$$Y_t = C_t + I_t \quad \Rightarrow \quad y_t = c_t + i_t$$

$\Rightarrow$ $$k_{t+1}(1+\eta) = (1-\delta)k_t + y_t - c_t$$

$\Rightarrow$ $$k_{t+1}(1+\eta) = (1-\delta)k_t + Ak_t{}^{\alpha} - c_t$$

Now, the optimization problem is transferred to

$$\max_{\{k_{t+1},c_t\}_{t=0}^{+\infty}} U_0 = \sum_{t=0}^{+\infty} \beta^t \frac{c_t^{1-1/\theta} - 1}{1-1/\theta}$$

s.t.

$$k_{t+1}(1+\eta) = (1-\delta)k_t + Ak_t{}^{\alpha} - c_t$$

$$c_t \geq 0, \ \ k_{t+1} \geq 0$$

while,

$$k_0 > 0 \ \ is \ \ given$$

Consider the medthod of Lagrange multiplier

$$F_0 = \sum_{t=0}^{+\infty} \beta^t \{ \frac{c_t^{1-\frac{1}{\theta}} - 1}{1-\frac{1}{\theta}} - \lambda_t [k_{t+1}(1+\eta) - (1-\delta)k_t - Ak_t{}^{\alpha} + c_t] \}$$

Calculate the first order condition

F.O.C.

$$\frac{\partial F_0}{\partial c_t} = 0$$

$$\Rightarrow \quad c_t^{-1/\theta} - \lambda_t = 0$$

$$\Rightarrow \quad \lambda_t = c_t^{-1/\theta}$$

$$\frac{\partial F_0}{\partial k_{t+1}} = 0$$

$$\Rightarrow \quad \beta^t[-\lambda_t(1+\eta)] + \beta^{t+1}[\lambda_{t+1}(1-\delta) + \lambda_{t+1}A \cdot \alpha k_{t+1}^{\alpha-1}] = 0$$

$$\Rightarrow \quad \frac{\lambda_t}{\lambda_{t+1}}(1+\eta) = \beta[1-\delta + A\alpha k_{t+1}^{\alpha-1}]$$

$$\Rightarrow \quad (\frac{c_{t+1}}{c_t})^{1/\theta}(1+\eta) = \beta[1-\delta + A\alpha k_{t+1}^{\alpha-1}]$$

$$\Rightarrow \quad \frac{1}{\theta}\ln\left(\frac{c_{t+1}}{c_t}\right) + \ln(1+\eta) = \ln\beta + \ln(1-\delta + A\alpha k_{t+1}^{\alpha-1})$$

Approximately, we have

$$\Delta c_{t+1} = \theta[ln\beta - \ln(1+\eta) + A\alpha k_{t+1}^{\alpha-1} - \delta]$$

Back to the status-transferring equation

$$k_{t+1}(1+\eta) = (1-\delta)k_t + A{k_t}^{\alpha} - c_t$$
$$\Delta k_{t+1} = \frac{1}{1+\eta}[A{k_t}^{\alpha} - c_t - (\eta+\delta)k_t]$$

Finally, we have two difference equation

$$\Delta c_{t+1} = \theta[ln\beta - \ln(1+\eta) + A\alpha k_{t+1}^{\alpha-1} - \delta]$$
$$\Delta k_{t+1} = \frac{1}{1+\eta}[A{k_t}^{\alpha} - c_t - (\eta+\delta)k_t]$$

Tranfer these two equation into continuous form

$$\frac{dc}{dt} = \theta[ln\beta - \ln(1+\eta) + A\alpha k^{\alpha-1} - \delta]$$
$$\frac{dk}{dt} = \frac{1}{1+\eta}[Ak^{\alpha} - c - (\eta+\delta)k]$$

■

These two equations form a two-dimension ODE system. c(t) and k(t) are non-negative function of t, which refer to consumption per capita and capital stock per capita respectively. $\theta, \beta, \eta, A, \alpha, \delta$ are all parameters waiting for calibration in the next section. Given c_0 and k_0, we are able to simulate the solution numerically.

Solution Analysis & Numerical Simulation

Caliberation

The following table shows numerical calibration of each parameter contained in the ODE system.

Parameter	Concept	Calibration
θ	Intertemporal Elasticity of Substitution (IES) / CRRA coefficient	0.74[2]
β	Discount factor	0.96
η	Population growth rate	0.005
A	Total Factor of Productivity (TFP)	1
α	Capital elasticity of substitution	0.3
δ	Capital depreciation rate	0.05

The estimation of IES follows Engelhardt and Kumar (2009). Other parameters are calibrated normally. Based on the calibration, we can simulate the ODE system above using both geometric methods and algorithm.

Geometric Analysis

We first analyze saddle point of the ODE system, which means the balance state of the economy. It can be solved by setting $\frac{dc}{dt} = 0$ and $\frac{dk}{dt} = 0$, and generates a constant solution.

$$\frac{dc}{dt} = \theta[ln\beta - \ln(1+\eta) + A\alpha k^{\alpha-1} - \delta] = 0$$

$$\frac{dk}{dt} = \frac{1}{1+\eta}[Ak^{\alpha} - c - (\eta+\delta)k] = 0$$

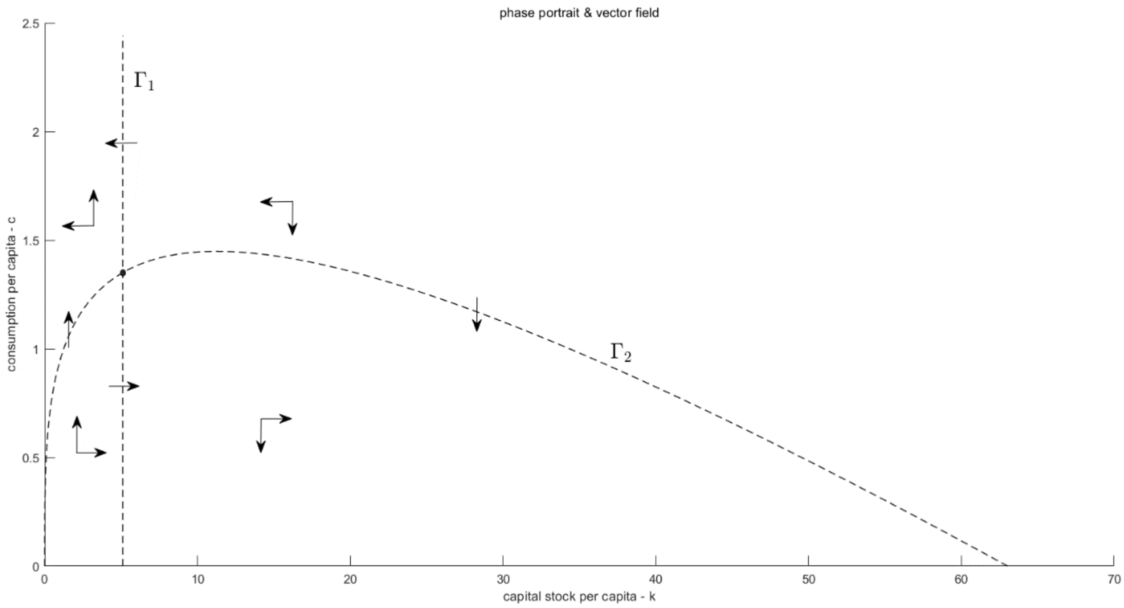
$$\Rightarrow \quad k^* = \left(\frac{A\alpha}{\delta + \ln(1+\eta) - ln\beta}\right)^{\frac{1}{1-\alpha}}$$

$$\Rightarrow \quad c^* = Ak^{*\alpha} - (\eta+\delta)k^*$$

Numerically, we have

$$k^* = 5.1069, \quad c^* = 1.3501$$

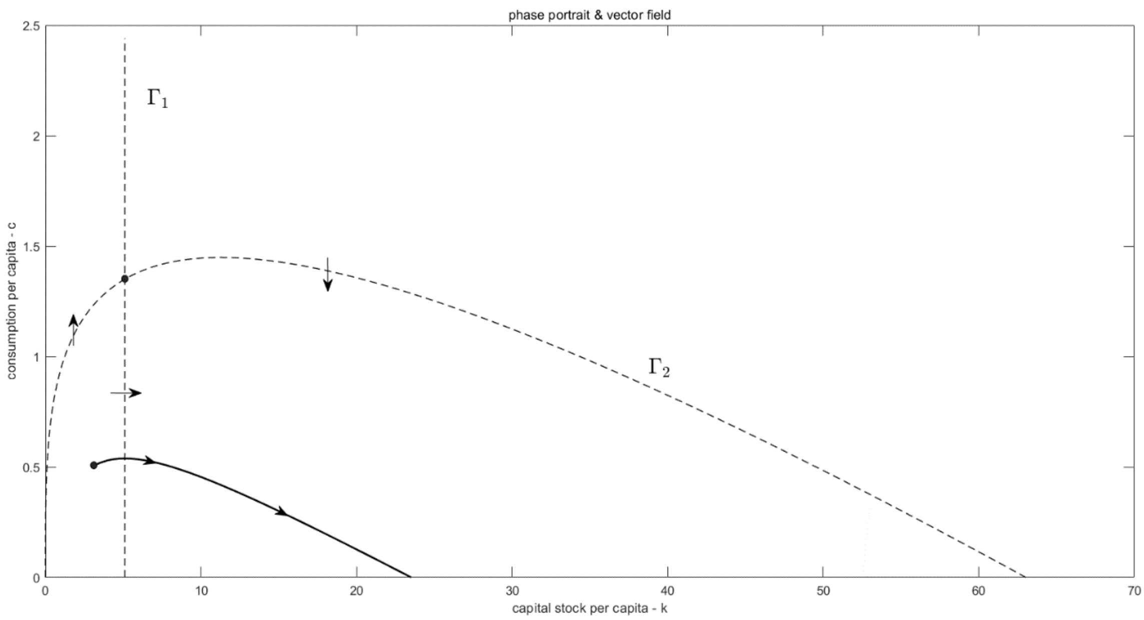
Notice that $\frac{dc}{dt} = 0$ and $\frac{dk}{dt} = 0$ are actually two curves that divide the 2-D diagram into four parts.



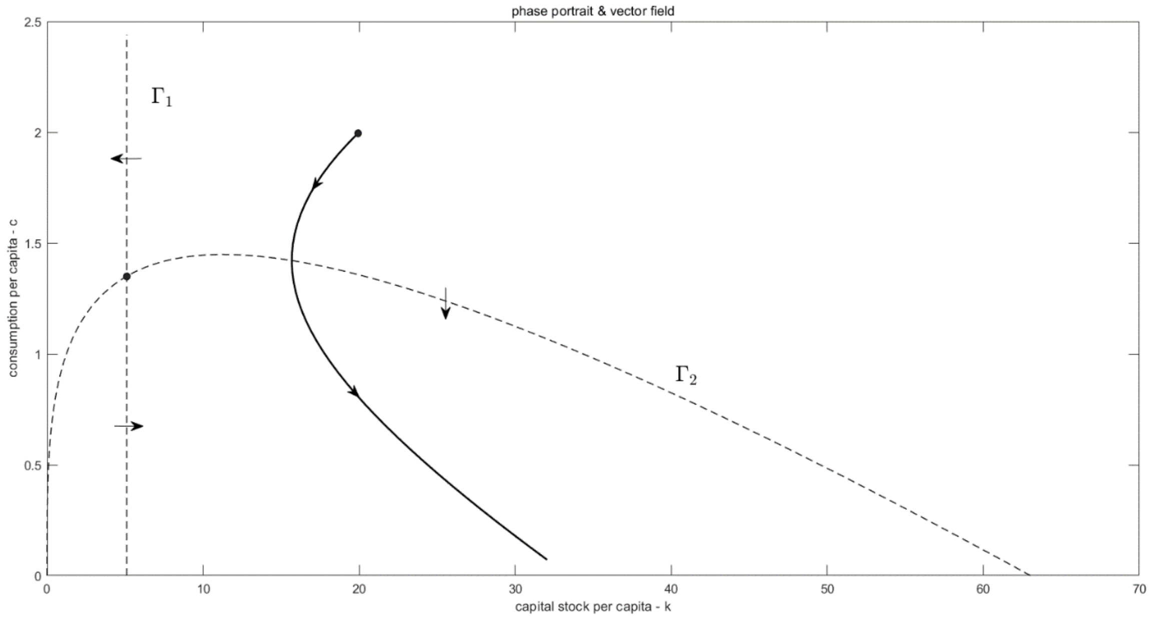
$$\Gamma_1:\ k = \left(\frac{A\alpha}{\delta + \ln(1+\eta) - \ln\beta}\right)^{\frac{1}{1-\alpha}}$$

$$\Gamma_2:\ \mathrm{c} = Ak^{\alpha} - (\eta + \delta)k$$

The intersection of Γ_1 and Γ_2 is exactly the saddle point $(k^*,\ c^*)$. When $(\mathrm{k},\ \mathrm{c})$ is below Γ_2, which means $\mathrm{c} < Ak^{\alpha} - (\eta + \delta)k$, we can deduce that $\frac{dk}{dt} > 0$. That is to say, the phase graph of solution is directed to the right when it's below Γ_2. Symmetrically, the solution path is directed to the left when it's above Γ_2. We can find the similar conclution based on Γ_1, $\frac{dc}{dt} > 0$ (upward) on the left of Γ_1 and $\frac{dc}{dt} < 0$ (downward) on the right of Γ_1.

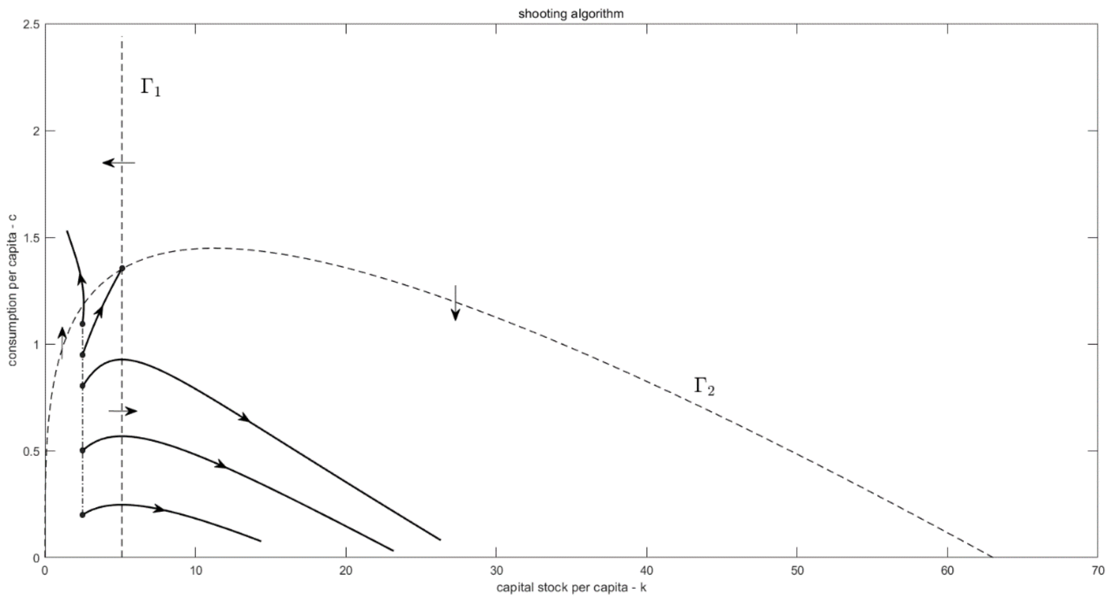


For example, in the case of setting $k_0 = 3,\ c_0 = 0.5$, the solution path first going upper right, since it's on the left of Γ_1 and below Γ_2. After walking through Γ_1, the path changes its direction and goes down right. Simmilarly, we can set $k_0 = 2,\ c_0 = 20$ and find the same pattern.



Shooting Algorithm

In macroeconomics, we are interested by how to direct an economy to the "stable status". That is to say, given k_0 as the initial status of the economy, we wants to find an appropriate c_0 that makes it possible for the social planner to drive its economy to the saddle point. In a specific case, we set $k_0 = 2.5$, trying to adjust c_0 that generates different solution path and directing it to be closer to the "stable status". This process feels like shooting a bullet from $k = 2.5$ targeting at the saddle point. That's why we call it "Shooting Algorithm".

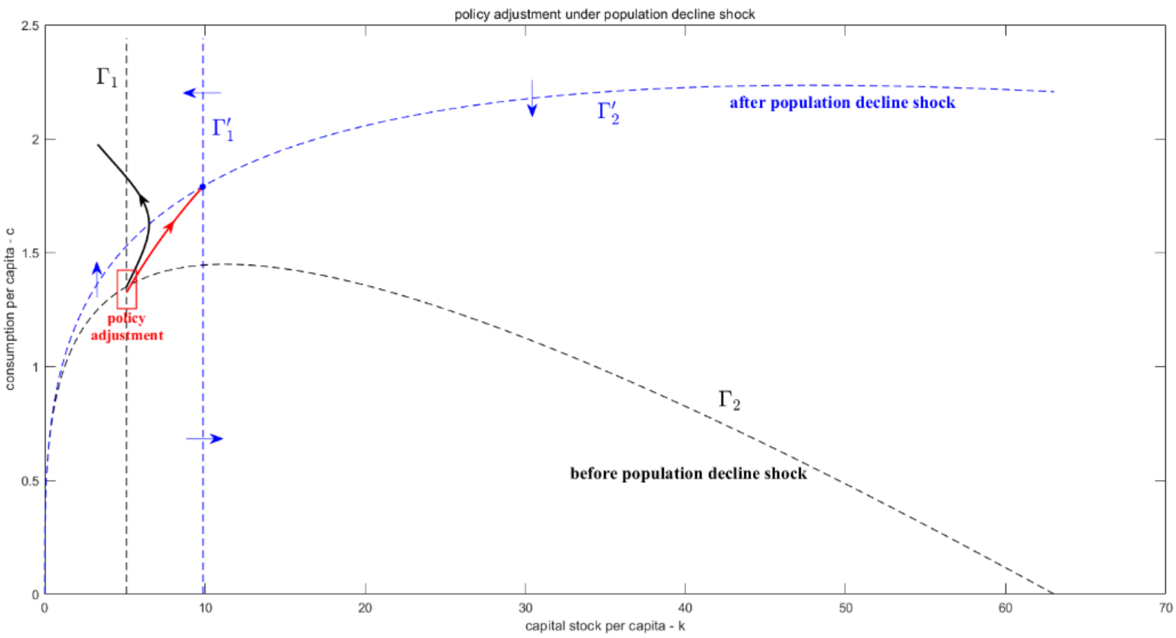


Finally, when setting $c_0^* = 0.9497$, we find the ideal solution path. If $c_0 > c_0^*$, the solution passes through Γ_2 and goes upper left. Simmilarly, if $c_0 < c_0^*$, the soltution goes down right far away from saddle point.

Population Decline

Here after we consider a population decline shock, which means that the population growth rate is exogenously reduced. For example, a famine hits the economy, or more realistically, the cost of raising up children becomes higher and higher and people's attitude toward reproduction changed in recent decade.

Specifically, we assume that the economy has already reached the stable status. After that, the growth rate drops from $\eta = 0.005$ to $\eta_{decline} = -0.03$. Both Γ_1 and Γ_2 will change simultaneously. Can the economy adjusts itself to stable status without policy intervention?



The answer is negative. As showed above, after population decline shock, if the economy remains its status at $(k^*, \ c^*)$, the solution path will depart from its ideal path towards new saddle point (the intersection of Γ_1' and Γ_2'). As a result, the social planner needs to change its original optimization plan, say exert a policy adjustment, to find the new stable status.

Conclusion

This study establishes a continuous Ramsey model under population decline by transforming a discrete-time framework into a two-dimensional ODE system. Through numerical simulations and phase diagram analysis, the calibrated model reveals that the economy converges to a saddle-point equilibrium under normal population growth. However, when subjected to a population decline shock (e.g., negative growth rates), the original equilibrium becomes unstable, necessitating policy interventions to guide the economy toward a new steady state. The shooting algorithm effectively identifies optimal initial consumption levels for stability, highlighting the model’s utility in analyzing dynamic adjustments. Future research could incorporate endogenous policy responses or additional demographic factors to enhance realism.

Appendix

MATLAB Code: Shooting Algorithm

```
% define parameters
theta = 0.74;
beta = 0.96;
eta = 0.005;
A = 1;
alpha = 0.3;
delta = 0.05;

% define ODE system
ramsey_ode = @(t, z) [
    theta * (log(beta) - log(1 + eta) + A * alpha * z(2)^(alpha - 1) -
delta);  % dc/dt
    (1/(1 + eta)) * (A * z(2)^alpha - z(1) - (eta + delta) *
z(2))            % dk/dt
];

% calculate fixed point;
k_ss = (A * alpha / (delta + log(1 + eta) - log(beta)))^(1/(1 - alpha));
c_ss = A * k_ss^alpha - (eta + delta) * k_ss;


% time span & initiation
t_span = [0 21];
c0 = 0.5;
k0 = 2.5;
% solve ODE
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);
% phase portrait & vector field
figure;
plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);
xlabel('capital stock per capita - k');
ylabel('consumption per capita - c');
title('phase portrait & vector field');
hold on;

% shooting 1
t_span = [0 34];
c0 = 0.8;
```

```
k0 = 2.5;
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);
plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);

% shooting 2
t_span = [0 10];
c0 = 0.2;
k0 = 2.5;
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);
plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);

% shooting 3
t_span = [0 8];
c0 = 1.1;
k0 = 2.5;
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);
plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);

% shooting 4
t_span = [0 50];
c0 = 0.9497;
k0 = 2.5;
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);
plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);




% shooting line
c_sh = 0.2 : 0.01 : 1.1;
k_sh = 2.5*ones(1,length(c_sh));
plot(k_sh, c_sh, 'k', 'LineWidth', 0.8, 'LineStyle','-.');

% stable line 1
k_1 = 0: 0.01 :(A/(eta+delta))^(1/(1-alpha));
c_1 = A*k_1.^alpha - (eta+delta)*k_1;
```

```
plot(k_1, c_1, 'k', 'LineWidth', 0.8, 'LineStyle','--');

% stable line 2
c_2 = 0 : 0.01 : max(c_1)+1;
k_2 = k_ss*ones(1,length(c_2));
plot(k_2, c_2, 'k', 'LineWidth', 0.8, 'LineStyle','--');
```

MATLAB Code: Population Decline

```
% define parameters
theta = 0.74;
beta = 0.96;
eta = 0.005;
eta_dec = -0.03;
A = 1;
alpha = 0.3;
delta = 0.05;

% define ODE system
ramsey_ode = @(t, z) [
    theta * (log(beta) - log(1 + eta_dec) + A * alpha * z(2)^(alpha - 1) -
delta);  % dc/dt
    (1/(1 + eta_dec)) * (A * z(2)^alpha - z(1) - (eta_dec + delta) *
z(2))             % dk/dt
];

% calculate fixed point;
k_ss = (A * alpha / (delta + log(1 + eta) - log(beta)))^(1/(1 - alpha));
c_ss = A * k_ss^alpha - (eta + delta) * k_ss;

% calculate fixed point of population decline case;
k_ss_dec = (A * alpha / (delta + log(1 + eta_dec) - log(beta)))^(1/(1 -
alpha));
c_ss_dec = A * k_ss_dec^alpha - (eta_dec + delta) * k_ss_dec;


figure;
% stable line 1
k_1 = 0: 0.01 :(A/(eta+delta))^(1/(1-alpha));
c_1 = A*k_1.^alpha - (eta+delta)*k_1;
plot(k_1, c_1, 'k', 'LineWidth', 0.8, 'LineStyle','--');
hold on;
```

```
% stable line 2
c_2 = 0 : 0.01 : max(c_1)+1;
k_2 = k_ss*ones(1,length(c_2));
plot(k_2, c_2, 'k', 'LineWidth', 0.8, 'LineStyle','--');


% stable line 3
k_3 = 0: 0.01 :(A/(eta+delta))^(1/(1-alpha));
c_3 = A*k_3.^alpha - (eta_dec+delta)*k_3;
plot(k_3, c_3, 'b', 'LineWidth', 0.8, 'LineStyle','--');
hold on;

% stable line 2
c_4 = 0 : 0.01 : max(c_1)+1;
k_4 = k_ss_dec*ones(1,length(c_4));
plot(k_4, c_4, 'b', 'LineWidth', 0.8, 'LineStyle','--');



% time span & initiation
t_span = [0 30];
c0 = c_ss;
k0 = k_ss;
% solve ODE
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);

plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);
xlabel('capital stock per capita - k');
ylabel('consumption per capita - c');
title('phase portrait & vector field');

% shooting 1
t_span = [0 100];
c0 = 1.324253;
k0 = k_ss;
[t, z] = ode45(ramsey_ode, t_span, [c0;k0]);
c_sol = z(:,1);
k_sol = z(:,2);
plot(k_sol, c_sol, 'k', 'LineWidth', 1.5);
```

Referrence

[1] Yang Yu. Lecture Notes: Advanced Macroeconomics I (ECON7001). Neoclassical growth model.
[2] Engelhardt, Gary V. and Anil Kumar (2009), "The Elasticity of Intertemporal Substitution: New Evidence from 401(k) Participation," Economics Letters 103 (1): 15-17.