Reducing Dimensions of Semidefinite Programs with Wedderburn-Artin Theorem

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Outline

Semidefinite Programs

Symmetry of Semidefinite Programs

Simplification by Wedderburn-Artin Theorem

Example: Lovász Number ϑ of Cycle Graph

Complex Semidefinite Program

$$\begin{aligned} \max \quad & \langle C, Y \rangle \\ \text{s.t.} \quad & \langle A_i, Y \rangle = b_i, \;\; i = 1, 2, \ldots, n \\ & Y \succeq 0 \end{aligned}$$

- $C, A_i \in \mathbb{C}^{X \times X}$ are given Hermitian matrices whose rows and columns are indexed by a finite set X. $(b_1, b_2, \ldots, b_n)^\top \in \mathbb{R}^n$ is a given vector.
- $Y \in \mathbb{C}^{X \times X}$ is a variable Hermitian matrix.
- $Y \succeq 0$ means that Y is positive semidefinite.
- $\langle C, Y \rangle = \sum_{i,j \in X} \overline{C_{ij}} Y_{ij} \in \mathbb{R}$ since C, Y are Hermitian.
- We assume that this program has an optimal solution.

Semidefinite Programming

- Semidefinite programming (SDP) has a wide range of applications:
  - combinatorial optimization, control theory...
- The dimensions of matrices causes difficulty for solving SDP.
  - It is crucial to exploit problems' structure to reduce the dimension.
- We introduce tools in representation theory to reduce the dimension of SDP when the problem has symmetry.
  - First we define the symmetry of SDP problems.
  - Second we explain how Wedderburn-Artin theorem reduces the dimension.
  - Finally we give an example of such simplification.

Outline

Group Action on Matrix

- Let G be a finite group acts on a finite set X.
- The group action extends naturally to $X \times X$:
  - For $a \in G, (x, y) \in X \times X$, define $a(x, y) := (ax, ay)$.
- The group action extends to $X \times X$-matrix:
  - For an $X \times X$-matrix M and $x, y \in X$, define $aM(x, y) := M(ax, ay)$.
  - M is invariant under G if $M = aM$ for all $a \in G$.

Symmetry of SDP

$$\begin{aligned} \max \quad & \langle C, Y\rangle \\ \text{s.t.} \quad & \langle A_i, Y\rangle = b_i, \; i = 1, 2, \ldots, n \\ & Y \succeq 0 \end{aligned}$$

The above SDP is invariant under G if:

- For every feasible solution Y and every $a \in G$, aY is also feasible;
- $\langle C, aY\rangle = \langle C, Y\rangle$ for all feasible Y and $a \in G$.

Suppose Y is an optimal solution,

- by convexity of SDP, $\frac{1}{|G|}\sum_{a\in G} aY$ is also an optimal solution.
- Such optimal solution is invariant under G.

Subspace of Matrices Invariant under Group Action

- Let $\mathcal{B}$ be the set of matrices which are invariant under G:
  - $\mathcal{B}$ forms a vector space over $\mathbb{C}$.
  - In fact, $\mathcal{B}$ is a semisimple algebra, which will be shown in next section.
- As discussed before, the SDP invariant under G always has an optimal solution invariant under G.
- It is equivalent to solve:

$$
\begin{aligned}
\max \quad & \langle C, Y\rangle \\
\text{s.t.} \quad & \langle A_i, Y\rangle = b_i, \ \ i = 1, 2, \ldots, n \\
& Y \succeq 0, \ \ Y \in \mathcal{B}
\end{aligned}
$$

Canonical Basis for $\mathcal{B}$

- By the action of G, $X \times X$ can be partitioned into the orbits: $R_1, R_2, \ldots, R_N$.
- For every R_r, we define the following matrix:

$$B_r \in \{0,1\}^{X \times X} \quad B_r(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_r \\ 0 & \text{otherwise.} \end{cases}$$

- $B_1, B_2, \ldots, B_N$ forms a canonical basis for $\mathcal{B}$
  - It is clear that $B_1, B_2, \ldots, B_N$ are linearly independent.
  - For each $M \in \mathcal{B}$ and each R_r, since M is invariant under G, we have

$$M(x, y) = M(x', y') \quad \forall (x, y), (x', y') \in R_r$$

First Step of Simplification

$B_1, B_2, \ldots, B_N$ forms a basis for $\mathcal{B}$

- For each $Y \in \mathcal{B}$, we have $Y = y_1 B_1 + \cdots + y_N B_N$ with $y_i \in \mathbb{C}$.
- Note that for each B_r, $B_r^\top$ is also an element of basis.
  - To make Y Hermitian, we ask $y_i = \overline{y}_j$ if $B_i = B_j^\top$.
- Let $c_r = \langle C, B_r \rangle$ and $a_{ir} = \langle A_i, B_r \rangle$, the SDP can be reformulated as follows:

$$
\begin{aligned}
\max \quad & c_1 y_1 + \cdots + c_N y_N \\
\text{s.t.} \quad & y_1, \ldots, y_N \in \mathbb{C}, \; y_i = \overline{y}_j \text{ if } B_i = B_j^\top \\
& a_{i1} y_1 + \cdots + a_{iN} y_N = b_i, i = 1, \ldots, n \\
& y_1 B_1 + \cdots + y_N B_N \succeq 0
\end{aligned}
$$

Outline

Main Idea of Simplification

The SDP is reformulated as follows

$$\begin{aligned} \max \quad & c_1 y_1 + \cdots + c_N y_N \\ \text{s.t.} \quad & y_1, \ldots, y_N \in \mathbb{C}, \;\; y_i = \overline{y}_j \text{ if } B_i = B_j^\top \\ & a_{i1} y_1 + \cdots + a_{iN} y_N = b_i, i = 1, \ldots, n \\ & y_1 B_1 + \cdots + y_N B_N \succeq 0 \end{aligned}$$

- In application, the dimension of $B_r \in \{0,1\}^{X \times X}$ could be large.
- For each $B_r \in \mathcal{B}$, we aim to reduce its dimension:
  - First, we show that $\mathcal{B}$ is a semisimple algebra.
  - Second, we use Wedderburn-Artin theorem for simplification.

Semisimple Algebra $\mathcal{B}$ and Jacobson Radical

- $\mathcal{B}$ is closed under matrix multiplication and has a basis $B_1, \ldots, B_N$:
  - $\mathcal{B}$ forms a finite-dimensional $\mathbb{C}$-algebra.

Theorem (Jacobson Radical)

Let A be a finite-dimensional F-algebra, then A is semisimple if and only if rad$(A) = 0$.

- Suppose that nonzero $M \in \text{rad}(\mathcal{B})$. By definition of $\mathcal{B}$, we have $M^* \in \mathcal{B}$.
- $\text{rad}(\mathcal{B})$ is a two-sided ideal, thus $M^* M \in \mathcal{B}$.
- $\text{rad}(\mathcal{B})$ is a nilpotent ideal, thus $M^* M$ is a nilpotent matrix.
- Therefore, all eigenvalues of $M^* M$ must be zero.
- As a result, $\text{trace}(M^* M) = \sum_{x,y \in X} |M(x, y)|^2 = 0$.
- This implies that $M = 0$. Contradicts.

Wedderburn-Artin Theorem

Theorem (Wedderburn-Artin)

Let A be a finite-dimensional semisimple algebra over a field F. Then,

$$A \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k)$$

where D_i is a finite-dimensional division algebra over the base field F.

If the base field F is algebraically closed (such as $\mathbb{C}$), there are no finite-dimensional division algebras over F other than F itself. Therefore,

$$A \cong M_{n_1}(F) \times M_{n_2}(F) \times \cdots \times M_{n_k}(F)$$

Block Diagonalization

- Applying Wedderburn-Artin theorem to $\mathcal{B}$, we know that there exists an isomorphism

$$\varphi : \mathcal{B} \to \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$$

- Isomorphism φ preserves eigenvalues hence positive semidefiniteness.

$$\begin{aligned} \max \quad & c_1 y_1 + \cdots + c_N y_N \\ \text{s.t.} \quad & y_1, \ldots, y_N \in \mathbb{C},\ \ y_i = \overline{y}_j \text{ if } B_i = B_j^\top \\ & a_{i1} y_1 + \cdots + a_{iN} y_N = b_i,\, i = 1, \ldots, n \\ & \boxed{y_1 \varphi(B_1) + \cdots + y_N \varphi(B_N) \succeq 0} \end{aligned}$$

- In many applications, $\sum_{i=1}^{k} n_i \ll |X|$.

Block Diagonalization

$$\begin{aligned} \max \quad & c_1 y_1 + \cdots + c_N y_N \\ \text{s.t.} \quad & y_1, \ldots, y_N \in \mathbb{C},\ \ y_i = \overline{y}_j \text{ if } B_i = B_j^\top \\ & a_{i1} y_1 + \cdots + a_{iN} y_N = b_i, i = 1, \ldots, n \\ & y_1 \varphi(B_1) + \cdots + y_N \varphi(B_N) \succeq 0 \end{aligned}$$

- There is an explicit construction of φ.
  - Such construction can be turned into an algorithm.
  - Symbolic calculation by hand is also possible.
- The process of construction leverages character theory and Maschke's theorem.
- We are going to use an example to demonstrate the simplification briefly.

Outline

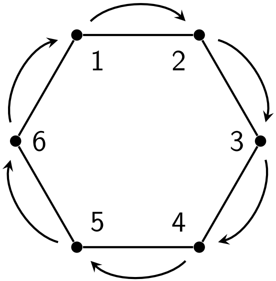
SDP Formulation for Lovász Number ϑ of C_n

- Consider the cycle graph $C_n = (V, E)$.
- The Lovász number ϑ of C_n has the following SDP formulation:

$$
\begin{aligned}
\max \quad & \sum_{i,j \in V} Y(i,j) \\
\text{s.t.} \quad & \operatorname{trace}(Y) = 1 \\
& Y(i,j) = Y(j,i) = 0 \quad \forall \{i,j\} \in E \\
& Y \succeq 0.
\end{aligned}
$$

- Here Y is a $V \times V$ matrix. When $|V|$ is large, solving this SDP is time-consuming.

Symmetry and Semisimple Algebra $\mathcal{B}$



- Consider a natural cyclic group G acts on V.
  - The left figure demonstrates a generator of G.
- G-action naturally extends to $V \times V$ matrix.
- For each $a \in G$ and each $V \times V$ matrix Y:
  - Define the corresponding basic cyclic permutation matrix P_a.
  - $aY = P_a Y P_a^\top$.
- It is clear that the SDP is invariant under G.

$$\begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

- If a $V \times V$ matrix Y is invariant under G,
  - then $Y = aY = P_a Y P_a^\top \quad \forall a \in G$.
  - Y must be a circulant matrix (left figure).
- Let $\mathcal{B}$ be the semisimple algebra defined by G.
  - The basis of $\mathcal{B}$ is exactly the set of basic cyclic permutation matrices $B_0, \ldots, B_{n-1}$.
- The basis is mutually commutative.
  - $\mathcal{B}$ is a commutative semisimple algebra.

Wedderburn-Artin Theorem and Isomorphism φ

- $\mathcal{B}$ is a finite-dimensional commutative semisimple algebra.
- By Wedderburn-Artin theorem, we have:

$$\mathcal{B} \cong M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times \cdots \times M_1(\mathbb{C})$$

- There exists an isomorphism φ diagonalizes all matrices in $\mathcal{B}$ simultaneously.
- We can construct a unitary matrix F such that
  - $\varphi(Y) = F^* Y F = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \quad \forall Y \in \mathcal{B}$.

Final Simplification: from SDP to LP

$$\begin{aligned} \max \quad & \sum_{i,j \in V} Y(i,j) \\ \text{s.t.} \quad & \operatorname{trace}(Y) = 1 \\ & Y(i,j) = Y(j,i) = 0 \quad \forall \{i,j\} \in E \\ & Y \succeq 0. \end{aligned}$$

- $Y = y_0 B_0 + y_1 B_1 + \cdots + y_{n-1} B_{n-1} \succeq 0 \;\Leftrightarrow\; \varphi(Y) \succeq 0.$
- $\varphi(B_r) = \operatorname{diag}(\lambda_{r0}, \lambda_{r1}, \ldots, \lambda_{r(n-1)})$

$$\operatorname{diag}\left(\sum_{r=0}^{n-1} y_r \lambda_{r0}, \ldots, \sum_{r=0}^{n-1} y_r \lambda_{r(n-1)}\right) \succeq 0 \;\Leftrightarrow\; \sum_{r=0}^{n-1} y_r \lambda_{ri} \geq 0 \quad i = 0, 1, \cdots, n-1$$