From Closed-Loop Integrals to Cyclic Monotonicity

Potential Energy and Revealed Preference Theory

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Abstract

This note develops cyclic monotonicity as the discrete counterpart of the closed-loop condition for a conservative force field.

- In physics, zero work around every closed curve makes path integrals independent of the route. Integrating from a fixed base point therefore recovers a potential, and hence potential energy.
- After a line integral is replaced by a finite path sum, cyclic monotonicity becomes the requirement that every discrete closed-loop integral be nonnegative. The same base-point construction—now using an infimum over finite paths—recovers a scalar function satisfying the discrete potential inequality.
- In revealed preference theory, suitably rescaled price–choice data form such a cyclically monotone discrete field. This is the step through which the discrete potential theorem supplies the utility function in Afriat's theorem.

1 The continuous model: force and potential energy

1.1 Work along a path

Let $X \subseteq \mathbb{R}^n$ be an open, path-connected region and let $F : X \to \mathbb{R}^n$ be a continuous force field. If $\gamma : [0,1] \to X$ is a piecewise C^1 path from s_1 to s_2, the work done by F along γ is

$$W_F(\gamma) := \int_\gamma F \cdot \mathrm{d}r = \int_0^1 \langle F(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t.$$

The basic question is whether this work depends only on the endpoints. The answer is the familiar closed-loop criterion.

Theorem 1.1 (Conservative field and potential)**.** *The following are equivalent:*

(i) for every piecewise C^1 closed curve $C \subset X$,

$$\oint_C F \cdot \mathrm{d}r = 0; \tag{1}$$

(ii) the integral of F between two points is independent of the path;

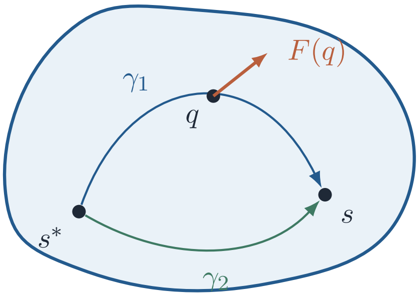
(iii) there exists $U : X \to \mathbb{R}$ such that, for every path γ from s_1 to s_2,

$$U(s_2) - U(s_1) = \int_\gamma F \cdot \mathrm{d}r. \tag{2}$$

The physical potential energy is $\Phi = -U$; thus

$$\Phi(s_2) - \Phi(s_1) = -\int_\gamma F \cdot \mathrm{d}r.$$

When U is differentiable, $F = \nabla U = -\nabla \Phi$.



$$\oint_{\gamma_1-\gamma_2} F \cdot \mathrm{d}r = 0$$

$$\Longrightarrow \quad \int_{\gamma_1} F \cdot \mathrm{d}r = \int_{\gamma_2} F \cdot \mathrm{d}r.$$

Figure 1: Two routes from the fixed point s^* to s. Their difference is a closed loop.

1.2 The base-point proof

Proof of Theorem 1.1. Assume first that every closed-curve integral is zero. If γ_1 and γ_2 both go from s_1 to s_2, follow γ_1 and then γ_2 in reverse. This gives a closed curve, so

$$0 = \oint_{\gamma_1-\gamma_2} F \cdot \mathrm{d}r = \int_{\gamma_1} F \cdot \mathrm{d}r - \int_{\gamma_2} F \cdot \mathrm{d}r.$$

Hence the integral depends only on the endpoints.

Now fix a base point $s^* \in X$. For each $s \in X$, choose any path γ_s from s^* to s and define

$$U(s) := \int_{\gamma_s} F \cdot \mathrm{d}r. \tag{3}$$

Path independence makes U well-defined. If η goes from s_1 to s_2, concatenate a path from s^* to s_1 with η. Then

$$U(s_2) = U(s_1) + \int_\eta F \cdot \mathrm{d}r,$$

which is (2). Conversely, summing (2) around any closed curve makes all endpoint values telescope, giving (1). □

The proof mechanism to remember. Fix s^*, assign to s the integral of the field along a path $s^* \to s$, and use the closed-loop condition to show that the assignment is consistent. The cyclic-monotonicity proof below is the same construction after a line integral has been replaced by a finite path sum.

2 Cyclic monotonicity as a discretized line integral

2.1 Finite paths and finite path integrals

Let $X \subseteq \mathbb{R}^n$ be nonempty and let $\rho : X \rightrightarrows \mathbb{R}^n$ be a set-valued field with $\rho(x) \neq \varnothing$. A finite path from x_1 to x_{m+1} consists of points $x_1, \dots, x_{m+1}$ and choices $z_i \in \rho(x_i)$. Define its discrete path integral by

$$I(P) := \sum_{i=1}^{m} \langle z_i, x_{i+1} - x_i \rangle . \tag{4}$$

This is the left-endpoint Riemann sum for a line integral. Indeed, if $z_i = F(x_i)$ and the points sample a smooth path, then

$$\sum_{i=1}^{m} \langle F(x_i), x_{i+1} - x_i \rangle \longrightarrow \int_\gamma F \cdot \mathrm{d}r$$

as the mesh becomes fine.

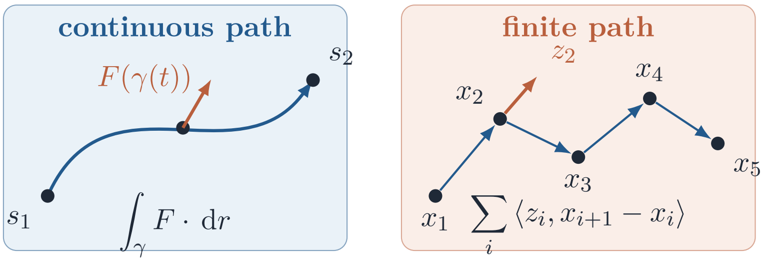


Figure 2: A finite path sum is the discrete counterpart of a line integral.

For a conservative field, an exact closed-loop integral is zero. In the finite setting we retain the one-sided no-cycle requirement

$$I(P) \geq 0 \qquad \text{for every closed finite path } P.$$

This loss of equality is important: discrete path values need not be independent of the route. Consequently, the scalar function will be defined by an *infimum* over all paths, rather than by the value of an arbitrary path.

2.2 Cyclic monotonicity as a discrete closed-loop condition

Definition 2.1 (Cyclic monotonicity)**.** The field ρ is *cyclically monotone* if, for every finite cycle

$$x_1, x_2, \ldots, x_m, x_{m+1} = x_1$$

and every choice $z_i \in \rho(x_i)$,

$$\sum_{i=1}^{m} \langle z_i, x_{i+1} - x_i \rangle \geq 0. \tag{5}$$

Thus cyclic monotonicity says exactly that every discrete closed-loop integral is nonnegative. This finite-path potential theorem is closely related to the treatments in [1, 3].

Theorem 2.2 (Discrete potential theorem)**.** *The field ρ is cyclically monotone if and only if there exists a function $u : X \to \mathbb{R}$ such that*

$$u(y) - u(x) \leq \langle z, y - x \rangle \qquad \textit{for all } x, y \in X \textit{ and } z \in \rho(x). \tag{6}$$

If X is convex, u can be chosen concave.

The inequality (6) is the discrete replacement for the energy identity (2): the one-edge path integral $\langle z, y - x \rangle$ is an upper bound on the change of the recovered scalar.

Proof. Assume that ρ is cyclically monotone. As in the physical proof, fix a base point $x^* \in X$; also fix $z^* \in \rho(x^*)$. For $x \in X$, let $\mathcal{P}(x)$ be the collection of finite paths

$$(x_1, z_1), (x_2, z_2), \ldots, (x_{M-1}, z_{M-1}), x_M$$

with $x_1 = x^*$, $z_1 = z^*$, $x_M = x$, and $z_i \in \rho(x_i)$. Define

$$u(x) := \inf_{P \in \mathcal{P}(x)} I(P) = \inf_{P \in \mathcal{P}(x)} \sum_{i=1}^{M-1} \langle z_i, x_{i+1} - x_i \rangle . \tag{7}$$

First, $u(x)$ is finite. The direct path $x^* \to x$ gives

$$u(x) \leq \langle z^*, x - x^* \rangle .$$

For the lower bound, take any path $P \in \mathcal{P}(x)$, choose $\bar{z} \in \rho(x)$, and close the path with the edge $x \to x^*$. Cyclic monotonicity gives

$$I(P) + \langle \bar{z}, x^* - x \rangle \geq 0,$$

hence $I(P) \geq \langle \bar{z}, x - x^* \rangle$. Therefore the infimum is not $-\infty$.

Next fix $z \in \rho(x)$. Take a path to x whose value is within ε of $u(x)$, and append the edge $(x, z) \to y$. This produces an admissible path to y, so

$$u(y) \leq u(x) + \varepsilon + \langle z, y - x \rangle .$$

Letting $\varepsilon \downarrow 0$ yields (6).

Conversely, suppose (6) holds. Apply it to every edge of a cycle and sum:

$$\sum_{i=1}^{m} (u(x_{i+1}) - u(x_i)) \leq \sum_{i=1}^{m} \langle z_i, x_{i+1} - x_i \rangle .$$

The left side telescopes to zero, proving cyclic monotonicity. Finally, (7) is an infimum of affine functions of the terminal point x; hence it is concave when X is convex. □

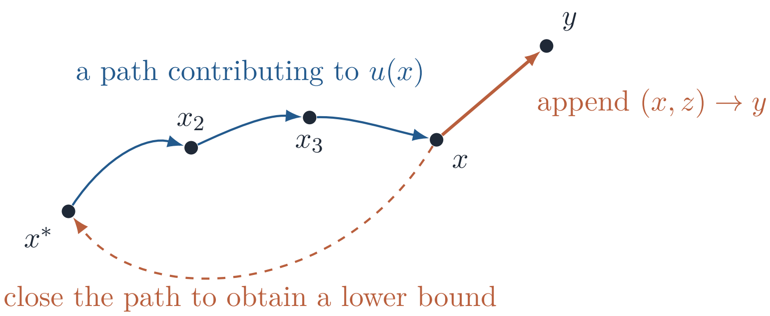


Figure 3: The discrete proof repeats the base-point argument from physics. Appending one edge gives the endpoint inequality; closing the path controls the infimum.

2.3 The two proofs side by side

	Continuous field	**Discrete field**
Path value	$\int_\gamma F \cdot \mathrm{d}r$	$I(P) = \sum_i \langle z_i, x_{i+1} - x_i \rangle$
Closed-loop rule	$\oint_C F \cdot \mathrm{d}r = 0$	$I(P) \geq 0$ for every closed P
Fixed point	choose s^*	choose x^* and $z^* \in \rho(x^*)$
Scalar recovery	$U(s) = \int_{s^*}^{s} F \cdot \mathrm{d}r$	$u(x) = \inf_{P: x^* \to x} I(P)$
Why it is valid	zero circulation gives exact path independence	the cycle inequality prevents arbitrarily cheap closed detours
Endpoint relation	$U(s_2) - U(s_1) = \int_{s_1}^{s_2} F \cdot \mathrm{d}r$	$u(y) - u(x) \leq \langle z, y - x \rangle$

Table 1: The main theorem is the finite-path analogue of potential recovery in physics.

3 Revealed preference as an application

3.1 Economic background

Consider finitely many price–choice observations

$$\mathcal{D} = \{(p^i, x^i)\}_{i=1}^N \subset \mathbb{R}_{++}^L \times \mathbb{R}_+^L.$$

For a systematic treatment of the revealed-preference framework, see [1]. At observation i, the consumer chooses x^i from the budget

$$B_i = \{x \in \mathbb{R}_+^L : p^i \cdot x \le p^i \cdot x^i\}.$$

A utility function $U : \mathbb{R}_+^L \to \mathbb{R}$ *rationalizes* the data if

$$x^i \in \arg\max_{x \in B_i} U(x) \qquad (i = 1, \dots, N).$$

If $p^i \cdot x^j \le p^i \cdot x^i$, then x^j was affordable when x^i was chosen, so x^i is directly revealed preferred to x^j. The Generalized Axiom of Revealed Preference (GARP) excludes a chain of such comparisons that returns with a strict contradiction.

For the direct connection with cyclic monotonicity, define

$$a_{ij} := p^i \cdot (x^j - x^i).$$

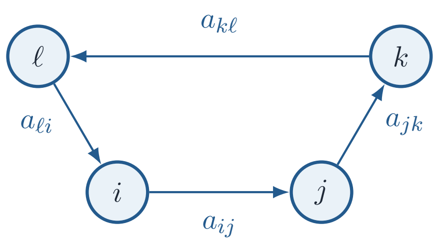
The data satisfy *additive GARP* (AGARP) if every index cycle $i_1, \dots, i_m, i_{m+1} = i_1$ satisfies

$$\sum_{r=1}^m a_{i_r i_{r+1}} = \sum_{r=1}^m p^{i_r} \cdot (x^{i_{r+1}} - x^{i_r}) \ge 0. \tag{8}$$

Thus AGARP is exactly cyclic monotonicity of the price correspondence

$$\rho_{\mathcal{D}}(x) := \{p^i : x^i = x\}.$$

For the original prices, AGARP is stronger than GARP. Afriat's key insight is that GARP allows each price vector to be rescaled by a positive number so that the resulting data satisfy AGARP.



$a_{ij} + a_{jk} + a_{k\ell} + a_{\ell i} \ge 0$

is the discrete closed-loop condition.

Figure 4: An expenditure cycle. AGARP requires nonnegative total discrete work around every such cycle.

3.2 Afriat's theorem and the role of cyclic monotonicity

Theorem 3.1 (Afriat [2])**.** *For the finite data set $\mathcal{D}$, the following are equivalent:*

(i) the data satisfy GARP;

(ii) there exist numbers $\phi_i \in \mathbb{R}$ and $\lambda_i > 0$ satisfying the Afriat inequalities

$$\phi_j - \phi_i \leq \lambda_i p^i \cdot (x^j - x^i) \qquad \textit{for all } i, j; \tag{9}$$

(iii) the data are rationalized by a continuous, increasing, concave utility function.

For the present notes, the important point is not a full proof of Afriat's theorem, but the following three-step route.

1. **Rescale prices to obtain a discrete no-cycle condition.** A finite rescaling lemma shows that GARP implies the existence of $\lambda_i > 0$ such that, for every cycle,
$$\sum_{r=1}^{m} \lambda_{i_r} p^{i_r} \cdot (x^{i_{r+1}} - x^{i_r}) \geq 0. \tag{10}$$
Equivalently, the correspondence
$$\rho_\lambda(x) := \{\lambda_i p^i : x^i = x\}$$
is cyclically monotone. This rescaling lemma is the genuinely finite-combinatorial part of Afriat's theorem.

2. **Apply the discrete potential theorem.** By Theorem 2.2, there is a function u on the observed bundles such that
$$u(x^j) - u(x^i) \leq \lambda_i p^i \cdot (x^j - x^i).$$
Setting $\phi_i := u(x^i)$ gives exactly the Afriat inequalities (9). Concretely, the values ϕ_i are obtained from the same fixed-base-point, infimum-over-paths construction as in (7).

3. **Extend the recovered values to a utility function.** Given (ϕ_i, λ_i), define
$$U(x) := \min_{1 \leq i \leq N} \{\phi_i + \lambda_i p^i \cdot (x - x^i)\}. \tag{11}$$
This function is continuous, increasing, and concave. The Afriat inequalities imply $U(x^i) = \phi_i$. If $x \in B_i$, then
$$U(x) \leq \phi_i + \lambda_i p^i \cdot (x - x^i) \leq \phi_i = U(x^i),$$
so x^i maximizes U on its observed budget.

$$\boxed{\begin{array}{c}\text{GARP} \Longrightarrow \text{positive price rescaling} \Longrightarrow \text{cyclic monotonicity} \\ \Longrightarrow \text{utility values} \Longrightarrow \text{rationalization}\end{array}}$$

4 One construction in three languages

	Physics	**Finite-path theory**	**Revealed preference**
Vector data	force $F(s)$	$z \in \rho(x)$	scaled price $\lambda_i p^i$ at x^i
Path quantity	$\int F \cdot \mathrm{d}r$	$\sum_i z_i \cdot (x_{i+1} - x_i)$	total scaled net expenditure along observations
Cycle rule	zero circulation	cyclic monotonicity	scaled AGARP
Fixed-point formula	$U(s) = \int_{s^*}^{s} F \cdot \mathrm{d}r$	$u(x) = \inf_{P: x^* \to x} I(P)$	$\phi_i = u(x^i)$
Recovered scalar	negative potential energy	discrete potential	utility levels, then U from (11)

Table 2: The same base-point construction in continuous, discrete, and economic language.

The conceptual direction is therefore

$$\text{closed-loop integral} \quad \longrightarrow \quad \text{discrete closed-loop sum} \quad \longrightarrow \quad \text{revealed-preference cycles.}$$

The main theorem should be read as a potential-recovery theorem for a discrete path integral. Its proof is natural once the corresponding proof for a conservative force field is kept in view.

References

[1] C. P. Chambers and F. Echenique, *Revealed Preference Theory*, Econometric Society Monographs, Cambridge University Press, 2016. doi:10.1017/CBO9781316104293.

[2] S. N. Afriat, *The Construction of a Utility Function from Expenditure Data*, International Economic Review 8 (1967), 67–77.

[3] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.