A Short Proof of the Four-Point Condition for Tree Metrics

Let X be a finite set with at least two elements, and let d be a metric on X. An *X-tree* is a finite tree T together with a map $\phi : X \to V(T)$ such that every vertex of degree at most two is labelled, meaning that it belongs to $\phi(X)$. We say that d is a *tree metric* if such a tree has strictly positive edge lengths and an injective label map satisfying

$$d(u, v) = d_T(\phi(u), \phi(v)) \qquad (u, v \in X),$$

where d_T denotes the length of the unique path between two points of the tree. Labels may lie at internal vertices; they need not be leaves.

We identify each label with its vertex when writing tree paths and distances. The notation $[u, v]$ means the unique path from u to v. A point inside an edge can be made a vertex by subdividing that edge, preserving its total length.

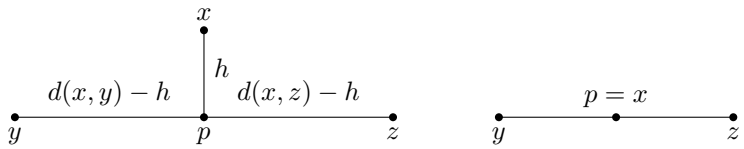
Theorem (Four-point condition). *The metric d of X is a tree metric if and only if, for every $i, j, k, l \in X$, the maximum of*

$$d(i, j) + d(k, l), \qquad d(i, k) + d(j, l), \qquad d(i, l) + d(j, k)$$

is attained at least twice.

Proof. Assume first that d satisfies the four-point condition. We construct a tree representing d by induction on $|X|$. If $|X| = 2$, join the two labelled vertices by an edge whose length is their distance. This gives the required tree.

For $|X| \geq 3$, choose *any* point $x \in X$. The restriction of d to $X \setminus \{x\}$ is still a metric satisfying the four-point condition. By induction, there is a tree T representing all distances between these remaining labels, with positive edge lengths, distinct labelled vertices, and every vertex of degree at most two labelled. Now, we must extend this tree to represent the distances involving x as well. The key idea is to make a **"projection"** from x to the tree.



$h > 0$: attach a new edge $\qquad$ $h = 0$: label p

1. Where should x be placed? We seek a point p of the existing tree at which to attach x by a new edge. If this edge has length h and p lies on $[y, z]$ for two remaining labels y, z, then the desired distances to y and z force

$$d(x, y) + d(x, z) - d(y, z) = 2h.$$

The two paths from x each use the new edge, while their portions along $[y, z]$ together have length $d(y, z)$. This suggests the following construction. Define

$$h = \frac{1}{2} \min_{\substack{u,v \in X \setminus \{x\} \\ u \neq v}} \big(d(x, u) + d(x, v) - d(u, v)\big), \tag{1}$$

and choose distinct y, z attaining the minimum. Such a pair exists because there are at least two remaining labels. The triangle inequality gives $h \geq 0$, and our choice gives

$$2h = d(x, y) + d(x, z) - d(y, z). \tag{2}$$

Place p on the existing path $[y, z]$ so that

$$d_T(y, p) = d(x, y) - h, \qquad d_T(p, z) = d(x, z) - h. \tag{3}$$

This position exists and is unique: $d_T(y, p) + d_T(p, z) = d(y, z)$ by (2), and both $d_T(y, p)$ and $d_T(p, z)$ are nonnegative by the triangle inequality. For example,

$$d_T(y, p) = d(x, y) - h = \frac{d(x, y) + d(y, z) - d(x, z)}{2} \geq 0.$$

If p is inside an edge, subdivide that edge at p. If $h > 0$, attach a new vertex labelled x to p by an edge of length h. If $h = 0$, label p itself by x, instead of adding a zero-length edge. Denote the resulting tree by T'.

2. Why does this placement give every required distance? After adding x, the paths between $X \setminus \{x\}$ still retain their lengths. The distances from x to y and z are also correct by (3). It remains to show that for every other $w \in X \setminus \{x, y, z\}$, $d(x, w) = h + d_T(p, w)$.

Indeed, after adding x, the new tree T' gives distance

$$d_{T'}(x, w) = h + d_T(p, w).$$

We will show that the four-point condition forces $d(x, w) = d_{T'}(x, w)$.

Because y, z minimize (1), we have

$$\begin{aligned} d(x, y) + d(x, w) - d(y, w) &\geq 2h, \\ d(x, z) + d(x, w) - d(z, w) &\geq 2h. \end{aligned}$$

Substitute (2) and rearrange:

$$\begin{aligned} d(x, w) + d(y, z) &\geq d(x, z) + d(y, w), \\ d(x, w) + d(y, z) &\geq d(x, y) + d(z, w). \end{aligned}$$

Thus the left-hand side is a largest four-point sum for x, w, y, z. The four-point condition says that it cannot be the unique largest sum. Therefore,

$$d(x, w) + d(y, z) = \max\{d(x, z) + d(y, w),\ d(x, y) + d(z, w)\}.$$

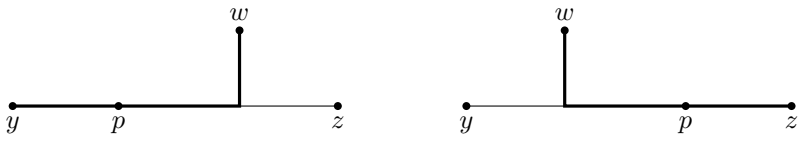
Subtract $d(y, z)$ and use (2) and (3). We obtain

$$\begin{aligned} d(x, w) &= \max\{d(y, w) + d(x, z) - d(y, z),\ d(z, w) + d(x, y) - d(y, z)\} \\ &= \max\{d(y, w) + 2h - d(x, y),\ d(z, w) + 2h - d(x, z)\} \\ &= h + \max\{d(y, w) - d_T(y, p),\ d(z, w) - d_T(z, p)\} \end{aligned} \tag{4}$$

To read the right-hand side geometrically, recall that T already represents the distances among y, z, w. The triangle inequality in T gives

$$\begin{aligned} d(y, w) - d_T(y, p) &\leq d_T(p, w), \\ d(z, w) - d_T(z, p) &\leq d_T(p, w). \end{aligned}$$

At least one is an equality, because at least one of the paths $[y, w]$ and $[z, w]$ passes through p. Indeed, as shown in the following figure, if p is an interior point of $[y, z]$, deleting p separates y and z into different components. Unless $w = p$, the point w lies in at most one of these components, so a path from the other endpoint to w must pass through p. If $w = p$, or if $p = y$ or $p = z$, the same assertion is immediate.



$[y, w]$ passes through p $[z, w]$ passes through p

Consequently, the maximum in (4) is exactly $d_T(p, w)$, and hence

$$d(x, w) = h + d_T(p, w) = d_{T'}(x, w).$$

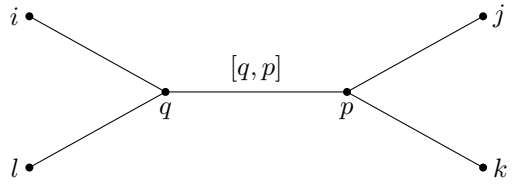
This proves that T' represents every distance in d.

It remains to check the requirements on a tree representation. Subdividing an edge and attaching a leaf preserve the tree property. All edge lengths are positive: a subdivision is made only at a strict interior point, and a new edge is added only when $h > 0$. The labels remain distinct when $h > 0$ because x is a new vertex. When $h = 0$, the point p cannot already carry an old label w, since the distance equality just proved would give $d(x, w) = 0$, contradicting that d is a metric.

Finally, every vertex of degree at most two is labelled. Old unlabelled vertices already had degree at least three, and their degrees do not decrease. A new subdivision vertex p has degree three if $h > 0$; if $h = 0$, it has degree two and is labelled by x. The only other possible new vertex is the labelled leaf x. Thus T' is an X-tree of precisely the required kind. This completes the induction.

Conversely, a tree metric satisfies the four-point condition. We include this direction to prove the full equivalence. If two labels coincide, we may reorder them so that $i = j$. The three sums are then $d(k, l)$ and two copies of $d(i, k) + d(i, l)$. The triangle inequality makes the repeated sum a maximum.

Now take four distinct labels in a representing tree. The paths joining i, j, k form a tripod, possibly with one arm of length zero: let p be the first point where the path from k to i meets $[i, j]$. The three arms are $[p, i]$, $[p, j]$, and $[p, k]$. Let q be the first point at which the path from l to p meets this tripod; take $q = l$ if l already lies on it. All paths from l to i, j, k pass through q, since a tree has a unique path between any two points. After permuting i, j, k, we may assume that $q \in [p, i]$.



Each line in the diagram represents a path; some may have length zero, including $[q, p]$. The pairing $(i, l), (j, k)$ uses each of the four outer paths once and does not use $[q, p]$. Each of the other two pairings uses the same outer paths once and $[q, p]$ twice. Adding their lengths therefore gives

$$\begin{aligned} d(i, j) + d(k, l) &= d(i, k) + d(j, l) \\ &= d(i, l) + d(j, k) + 2d_T(p, q). \end{aligned}$$

Since $d_T(p, q) \geq 0$, the first two sums are equal maxima. Reordering the labels merely permutes the three sums, so the four-point condition holds in the original order as well. □