Shapley-Folkman Lemma in Economics

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May 2026

Outline

Consumer's Decision Environment

Consider a consumer making the following purchase decision.

- $x \in X = \mathbb{R}^L_+$ denotes a consumption bundle (a "basket" of goods) over L different goods.
- $p = (p_1, \cdots, p_L) \in \mathbb{R}^L_{++}$ denotes the price vector.
- $w \in \mathbb{R}_+$ denotes the wealth.
- Consumer chooses bundles from a budget set

$$B_{p,w} := \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}.$$

  - any consumption bundle $x \notin B_{p,w}$ is unaffordable.

Walrasian Demand Correspondence

In principle, a consumer may choose one bundle, multiples bundles, or nothing.

- In order to depict how the consumer makes decisions, we need a tool to describe the mapping from parameters (p, w) to a set.

Definition (Walrasian Demand)

Given a utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$, the Walrasian demand (correspondence) $x : \mathbb{R}^L_{++} \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^L_+$ is defined by

$$x(p, w) := \arg\max_{x \in \mathbb{R}^L_+} u(x) \quad \text{s.t. } p \cdot x \leq w.$$

Assumptions on Utility Functions

Proposition 1

If u is continuous, then $x(p, w)$ is nonempty-valued.

- $B_{p,w}$ is compact.

Proposition 2

If u is strictly increasing, then

$$p \cdot x = w, \ \forall x \in x(p, w)$$

- consumer always uses up her income.

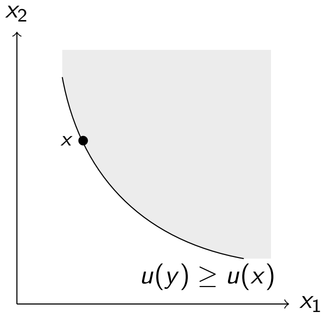
Assumptions on Utility Functions

Definition (Quasi-concave)

A function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is quasi-concave if $u(x) \geq u(y)$ implies

$$u(\alpha x + (1-\alpha)y) \geq u(y), \ \forall \alpha \in (0,1)$$

or equivalently, for all $x \in \mathbb{R}^L_+$, the set of "weakly better than x" (upper contour set) $\{y \in \mathbb{R}^L_+ : u(y) \geq u(x)\}$ is convex.



Assumptions on Utility Functions

Proposition 3

If u is quasi-concave, then $x(p, w)$ is convex-valued.

Definition (Strictly Quasi-concave)

A function $u : \mathbb{R}^L_+ \to \mathbb{R}$ is quasi-concave if $u(x) \geq u(y)$ and $x \neq y$ imply

$$u(\alpha x + (1-\alpha)y) > u(y),\ \forall \alpha \in (0,1)$$

Proposition 4

If u is strictly quasi-concave, then $|x(p, w)| \leq 1$.

We usually assume that the utility function is continuous, strictly increasing, and strictly quasi-concave, thus $x(p, w)$ is single-valued and continuous.

Outline

Pure Exchange Economy

We study how goods are traded between consumers in an exchange economy:

- L goods: $i \in \{1, 2, \cdots, L\}$
- N consumers: $j \in \{1, 2, \cdots, N\}$, where each consumer j has
  - a utility function $u^j(x^j)$, where $x^j = (x_1^j, \cdots, x_L^j)$ is j's consumption bundle.
  - an endowment $e^j = (e_1^j, \cdots, e_L^j) \in \mathbb{R}_+^L \setminus \{0\}$ that describes what j has before exchanging.

Assumption: the aggregate endowment of each good is strictly positive, otherwise we can delete this good from our economy

$$\sum_{j=1}^{N} e^j \gg 0.$$

General Equilibrium

Definition (Allocation and Feasibility)

An allocation (of resources) is a matrix

$$X = (x^1, x^2, \cdots, x^N) \in \mathbb{R}_+^{L \times N}.$$

An allocation X is feasible if

$$\sum_{j=1}^{N} x^j \leq \sum_{j=1}^{N} e^j.$$

General Equilibrium

Definition (Walrasian Equilibrium)

A Walrasian (General) Equilibrium $(\bar{X}, \bar{p})$ consists of an allocation $\bar{X} = (\bar{x}^1, \cdots, \bar{x}^N)$ and a price vector $\bar{p} = (\bar{p}_1, \cdots, \bar{p}_L)$ satisfying

1. consumer-optimality: for each j,

$$\bar{x}^j \in \arg\max_{x^j \in \mathbb{R}^L_+} u^j(x^j) \quad \text{s.t. } \bar{p} \cdot x^j \leq \bar{p} \cdot w^j$$

2. feasibility (market clear): $\bar{X}$ is feasible.

A basic and important question for economists:

- Does the General Equilibrium always exist?
- Under some assumptions, the answer is Yes! (Arrow and Debreu, 1954)

Aggregate Excess Demand

To prove the existence of an equilibrium, it is convenient to work with the excess demand function.

Definition (Excess Demand)

Let $x^j(p, p \cdot e^j)$ be the Walrasian demand of consumer j. The individual excess demand is $z^j(p) = x^j(p, p \cdot e^j) - \{e^j\}$. The aggregate excess demand is

$$z(p) = \sum_{j=1}^{N} z^j(p) \quad \text{(Minkowski Sum)}$$

An equilibrium price vector p^* is simply one that clears the market:

$$z(p^*) \cap \mathbb{R}^L_- \neq \emptyset$$

Properties of Aggregate Excess Demand

Assuming utilities are continuous, strictly increasing, and strictly quasi-concave, $z(p)$ satisfies:

1. $z(\cdot)$ is single-valued.
2. **Continuity**: $z(\cdot)$ is continuous on $\mathbb{R}^L_{++}$.
3. **Homogeneity of degree zero**: $z(\lambda p) = z(p)$ for any $\lambda > 0$. (Scaling prices does not change the budget set).
4. **Walras's Law**: For any $p \gg 0$,

$$p \cdot z(p) = 0$$

Proof of 3: By Proposition 2, $p \cdot x^j = p \cdot e^j$ for all j. Summing over j gives $p \cdot \sum x^j = p \cdot \sum e^j$, which implies $p \cdot z(p) = 0$.

The Price Simplex

Because $z(p)$ is homogeneous of degree zero, we can normalize prices without loss of generality.

We restrict our search for an equilibrium price to the unit simplex:

$$\Delta = \left\{ p \in \mathbb{R}^L_+ : \sum_{i=1}^{L} p_i = 1 \right\}$$

Notice that Δ is a **non-empty, compact, and convex** subset of $\mathbb{R}^L$. This topological property is crucial for applying fixed-point theorems.

Brouwer's Fixed Point Theorem

To prove the existence of an equilibrium $p^* \in \Delta$, we rely on a fundamental result from topology.

Brouwer's Fixed Point Theorem

Let $S \subset \mathbb{R}^n$ be a non-empty, compact, and convex set. If $f : S \to S$ is a continuous function, then there exists at least one fixed point $x^* \in S$ such that

$$f(x^*) = x^*.$$

Our goal: Construct a continuous mapping $f : \Delta \to \Delta$ such that its fixed point corresponds to a market-clearing price vector.

Existence of General Equilibrium

Define a "Walrasian auctioneer" mapping $f : \Delta \to \Delta$ by:

$$f_i(p) = \frac{p_i + \max\{0, z_i(p)\}}{1 + \sum_{k=1}^{L} \max\{0, z_k(p)\}} \quad \text{for } i = 1, \ldots, L.$$

- **Intuition**: If there is excess demand for good i ($z_i > 0$), the auctioneer raises its relative price. If there is excess supply ($z_i \leq 0$), the price falls (relative to goods in excess demand).
- **Well-defined**: $f_i(p) \geq 0$ and $\sum_{i=1}^{L} f_i(p) = 1$, so $f(p) \in \Delta$.
- **Continuous**: Since $z(p)$ and $\max\{0, \cdot\}$ are continuous, f is continuous on Δ.

Existence of General Equilibrium

By Brouwer's Theorem, there exists a fixed point $p^* \in \Delta$ such that $f(p^*) = p^*$. Thus, for all i:

$$p_i^* = \frac{p_i^* + \max\{0, z_i(p^*)\}}{1 + \sum_{k=1}^{L} \max\{0, z_k(p^*)\}}$$

Multiply both sides by $z_i(p^*)$ and sum over all i:

$$\sum_{i=1}^{L} p_i^* z_i(p^*) = \frac{\sum_{i=1}^{L} p_i^* z_i(p^*) + \sum_{i=1}^{L} z_i(p^*) \max\{0, z_i(p^*)\}}{1 + \sum_{k=1}^{L} \max\{0, z_k(p^*)\}}$$

By Walras's Law, $\sum p_i^* z_i(p^*) = 0$. This forces:

$$\sum_{i=1}^{L} z_i(p^*) \max\{0, z_i(p^*)\} = 0$$

Existence of General Equilibrium

$$\sum_{i=1}^{L} z_i(p^*) \max\{0, z_i(p^*)\} = 0$$

Notice that every term in this sum is non-negative, because $z_i \max\{0, z_i\} = z_i^2 \geq 0$ if $z_i > 0$, and equals 0 if $z_i \leq 0$.

Therefore, we must have $z_i(p^*) \max\{0, z_i(p^*)\} = 0$ for all i.

- This implies $z_i(p^*) \leq 0$ for all i, thus $z(p^*) \cap \mathbb{R}^L_- \neq \emptyset$.
- Therefore, p^* clears all markets.

Conclusion: A Walrasian (General) Equilibrium exists!

Outline

Problems of Convexity Assumptions

- In the previous proof, we assumed utility functions are strictly quasi-concave to guarantee a single-valued, continuous demand function.
- Kakutani's fixed-point theorem relaxes this to quasi-concave utilities, allowing for multi-valued but convex-valued demand correspondences.
- **But this is not enough in reality!**
  - Many goods are strictly **indivisible** (e.g., cars, machines, houses).
  - Consumption sets become discrete, destroying convexity.
  - Fixed-point theorems completely fail.
  - Does General Equilibrium still exist?

Back to Convex Geometry

Before introducing the non-convex economy, let us look at a purely mathematical result regarding the Minkowski sum of sets.

Let $S_1, S_2, \cdots, S_N$ be arbitrary non-empty subsets of $\mathbb{R}^L$.

- They are **not** required to be convex.
- They are **not** required to be connected or closed.

Let $S = \sum_{j=1}^{N} S_j$ be their Minkowski sum.
The convex hull operator is linear with respect to the Minkowski sum:

$$\mathrm{conv}(S) = \mathrm{conv}\left(\sum_{j=1}^{N} S_j\right) = \sum_{j=1}^{N} \mathrm{conv}(S_j)$$

The Shapley-Folkman Lemma (1969)

Shapley-Folkman Lemma

For any point $x \in \text{conv}(S)$, there exists a representation:

$$x = \sum_{j=1}^{N} y^j$$

where $y^j \in \text{conv}(S_j)$ for all $j = 1, \ldots, N$, such that the number of elements with $y^j \notin S_j$ is **at most** L.

In other words, we can find an index set $I \subset \{1, 2, \cdots, N\}$ such that:

1. $|I| \leq L$ (The dimension of the space).
2. $y^j \in S_j$ for all $j \notin I$.

Intuition of Shapley-Folkman Lemma

- It is a powerful extension of **Carathéodory's Theorem**.
- It states that when we add many non-convex sets together, the "non-convexity" of the sum does not grow with the number of sets N.
- Instead, the non-convexity is strictly bounded by the dimension of the space L.
- **Key Insight:** If $N \gg L$, the sum S is "almost" convex!
- Conjecture: If the number of consumers is large enough, the economy may have an approximate General Equilibrium, even without convexity assumptions.

Now, let's bring this powerful geometric tool back to economics.

Settings of the Non-convex Economy

Let's formalize the real, non-convex economy.

For each consumer $j \in \{1, \cdots, N\}$:

- **Consumption Set:** $X^j \subset \mathbb{R}_+^L$ is closed, but not necessarily convex. (e.g., $X^j = \mathbb{Z}_+ \times \mathbb{R}_+$ for indivisible and divisible goods).
- **Endowment:** $e^j \in X^j \setminus \{0\}$.
- **Utility Function:** $u^j : X^j \to \mathbb{R}$ is continuous, but not necessarily quasi-concave.

The Real Excess Demand Correspondence

Given a price $p \in \Delta$, the budget set is $B^j(p) = \{x \in X^j : p \cdot x \leq p \cdot e^j\}$.

Definition (Real Excess Demand)

The real excess demand correspondence for consumer j is:

$$z^j(p) = \arg\max_{x \in B^j(p)} u^j(x) - \{e^j\}$$

By Berge's Maximum Theorem, $z^j(p)$ is Upper Hemi-Continuous (UHC) and compact-valued.

However, because X^j and upper contour sets of u^j lack convexity, $z^j(p)$ is **not convex-valued**.

The Fictitious Convexified Economy

To restore the topological properties required for fixed-point theorems, we construct a fictitious economy.

Definition (Convexified Excess Demand)

We apply the convex hull operator to individual demand:

$$\hat{z}^j(p) = \text{conv}(z^j(p))$$

The aggregate convexified excess demand is:

$$\hat{z}(p) = \sum_{j=1}^{N} \hat{z}^j(p) = \sum_{j=1}^{N} \text{conv}(z^j(p))$$

Properties of the Fictitious Economy

A crucial topological fact: the convex hull operator preserves UHC and compactness.

Therefore, the convexified correspondence $\hat{z}(p)$:

- is Upper Hemi-Continuous (UHC).
- is compact-valued.
- is **convex-valued** (by construction).

Furthermore, Walras's Law (in inequality form) still holds:

$$p \cdot \hat{y} \leq 0, \quad \forall \hat{y} \in \hat{z}(p)$$

because every point in the convex hull is a convex combination of points that satisfy the budget constraint.

The Pseudo-Equilibrium

Since $\hat{z}(p)$ satisfies all the prerequisites of Kakutani's Fixed Point Theorem, by similar constructions, a pseudo-equilibrium is guaranteed to exist.

Existence of Pseudo-Equilibrium

There exists a price vector $p^* \in \Delta$ and an aggregate excess demand vector $\hat{z}^* \in \hat{z}(p^*)$ such that:

$$\hat{z}^* \leq 0$$

- The market clears! But wait...
- $\hat{z}^*$ is an element of $\sum \text{conv}(z^j(p^*))$, instead of $\sum z^j(p^*)$.
- It might be formed by fictitious bundles that consumers would never actually choose in reality (e.g., buying 0.5 cars).

Applying the Shapley-Folkman Lemma

We have the pseudo-equilibrium clearing the market:

$$\hat{z}^* \in \sum_{j=1}^{N} \mathrm{conv}(z^j(p^*))$$

Let $S_j = z^j(p^*)$ be the real choice sets.

By the Shapley-Folkman Lemma, there exists a decomposition:

$$\hat{z}^* = \sum_{j=1}^{N} y^j, \quad \text{where } y^j \in \mathrm{conv}(S_j)$$

and an index set I with $|I| \leq L$, such that $y^j \in S_j$ for all $j \notin I$.

Decomposition of Choices

Let's look at the decomposition of the clearing demand $\hat{z}^*$:

$$\underbrace{\hat{z}^*}_{\text{Clears the market}} = \underbrace{\sum_{j \notin I} y^j}_{\text{Real Choices}} + \underbrace{\sum_{j \in I} y^j}_{\text{Fictitious Choices}}$$

- **Group 1 ($j \notin I$):** These $N - |I|$ consumers are making **true** optimal choices that maximize their non-convex utilities.
- **Group 2 ($j \in I$):** These consumers are assigned bundles in the convex hull.
- **The Magic Bound:** The size of Group 2 is bounded by $|I| \leq L$.

Return to Reality

We cannot allow any fictitious choices in a real economy.

Therefore, we need to force Group 2 back to reality by constructing the following allocation:

- For $j \notin I$, keep their real choices: $x^j = y^j \in S_j$.
- For $j \in I$, arbitrarily select a real optimal bundle: $x^j \in S_j$ to replace the fictitious y^j.

Now, define the **True Aggregate Excess Demand** at price p^*:

$$z^* = \sum_{j=1}^{N} x^j$$

Note that z^* is completely formed by real, utility-maximizing choices.

Bounding the Market Shortage

How far is the real market z^* from the perfectly clearing fictitious market $\hat{z}^*$?

$$\|z^* - \hat{z}^*\| = \left\| \sum_{j=1}^{N} x^j - \sum_{j=1}^{N} y^j \right\| = \left\| \sum_{j \in I} (x^j - y^j) \right\| \leq \sum_{j \in I} \|x^j - y^j\|$$

Let M be the maximum "radius of non-convexity" among all consumers (e.g., the physical size of one indivisible car). Thus $\|x^j - y^j\| \leq M$.
Since $|I| \leq L$:

$$\|z^* - \hat{z}^*\| \leq L \cdot M$$

Ross Starr's Theorem (1969)

Since the fictitious economy cleared ($\hat{z}^* \leq 0$), the absolute shortage in the real economy is strictly bounded:

$$\max\{0, z^*\} \leq L \cdot M$$

Now, consider the **per-capita** market shortage:

$$\frac{1}{N} \max\{0, z^*\} \leq \frac{L \cdot M}{N}$$

If we consider a sequence of economies where the number of consumers $N \to \infty$ (while L and M remain fixed):

$$\lim_{N \to \infty} \frac{L \cdot M}{N} = 0$$

Conclusion: In a sufficiently large economy, market clearing holds approximately. **Approximate General Equilibrium exists!**